A Generalized Model for the Prediction of Controller Intervention Rates in the En Route Air Traffic Control System

by

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A GENERALIZED MODEL FOR THE PREDICTION OF CONTROLLER INTERVENTION RATES IN THE EN ROUTE AIR TRAFFIC CONTROL SYSTEM

by

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ABSTRACT

A generalized model of the domestic en route air traffic control system is constructed, which can be used to predict the rate at which controllers will need to intervene in the flow of radar-controlled traffic to prevent the violation of minimum horizontal separation standards. The model considers both crossing and overtaking conflicts, and includes both on- and off-airway traffic. Further, the model is able to incorporate complex airway intersections, including those involving more than two crossing airways, as well as those which permit aircraft to change airways at the intersection. A delayed negative exponential distribution on aircraft interarrival distances is used to reflect the traffic separation efforts of controllers in neighboring sectors.

The model is initially presented in a two-dimensional form and then extended to three dimensions by the use of traffic "sources" and "sinks" to represent climbing/descending aircraft which appear and then disappear in successive flight levels. The three-dimensional version is not extended to Terminal Control Areas, due to the decidedly non-random traffic flow into and out of large airports.

Monte Carlo simulations of simple airway intersections and single airway segment overtaking situations are used to confirm the expected intervention rates predicted by the analytic model. The simulations also serve to illustrate the significant variation to be expected about the mean intervention rate. An expression for the variance about the mean intervention rate at simple intersections is derived by conditioning on the actual (as opposed to expected) traffic density and then using well known results for the variance of binomial random variables.

An extensive sensitivity analysis is performed on several of the model's key assumptions. The model is found to be particularly sensitive to assumptions about steady state behavior in mean traffic flow rates and somewhat less sensitive to errors in aircraft velocity distributions.

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Chapter 1

1.1 Background

The air traffic control system in the United States is a highly complex structure designed to meet several often conflicting objectives. A primary concern is safety, so the system is designed to provide protection against aircraft collisions as well as various weather hazards like thunderstorms, turbulence, and icing. In addition, the Federal Aviation Administration (FAA), which administers the domestic air traffic control system, sets minimum standards for personnel (eg. pilots, air traffic controllers) through standardized training and periodic testing. The FAA also defines minimum standards for both equipment (aircraft) and facilities (airports, aircraft maintenance shops) to insure that necessary hardware is both available and functioning properly.

Another major objective of the domestic air traffic control system is to encourage an expeditious and orderly flow of traffic from origin to destination, with minimum ground and (more importantly), inflight delays. The same users of the system who insist upon "maximum possible safety" also want to minimize disruption to the flow of scheduled traffic; often these two objectives are in conflict. As a simple example, consider the relationship between the minimum allowable separation between inflight aircraft, and the overall capacity of the system. Current FAA rules specify minimum horizontal and vertical separation standards as a function of the flight regime (eg. takeoff/landing, low or high altitude cruise) as well as the type of aircraft involved. When the overall system or portions thereof operate near capacity - as is often the case - unavoidable delays are generated. Often one hears suggestions that a reduction in one or more of the minimum separation standards would by a good way to increase system capacity and thus reduce unscheduled delays. Unfortunately, such reductions would probably have a negative impact on flight safety - a negative impact that is often difficult to quantify. Conversely, any attempt to increase the system safety level by increasing minimum separation requirements will tend to reduce system capacity and thus increase the number and/or magnitude of unscheduled delays.

Another complicating factor that must be dealt with by the air traffic control system is the existence of two different types of traffic: that which adheres to often complex Instrument Flight Rules (IFR) designed to support a regular flow of traffic at night and during adverse weather, and that governed by less demanding Visual Flight Rules (VFR), which are used at lower altitudes during daylight and good weather. Aircraft flying under Instrument Flight Rules (so-called "IFR traffic") must have complex and expensive navigation and communication equipment onboard in order to comply with FAA standards, while the requirements for aircraft flying under VFR are substantially less. Commercial airliners and most military aircraft usually fly under IFR, while smaller, less expensively equipped private aircraft (eg. Cessna 150's, Piper Cubs) tend to use VFR.

Another major distinction between VFR and IFR traffic is that the former has a great deal of latitude in determining the route flown between origin and destination; often, no route at all is specified. IFR traffic, on the other hand, must declare a specific intended route of flight. Furthermore, such flights are strongly encouraged by the FAA to make use of published "highways in the sky," flight paths defined by periodic radio navigation aids to facilitate an orderly flow of high-density traffic.

A typical highway in the sky, or airway, is shown in Figure 1-1. The airway has a specified

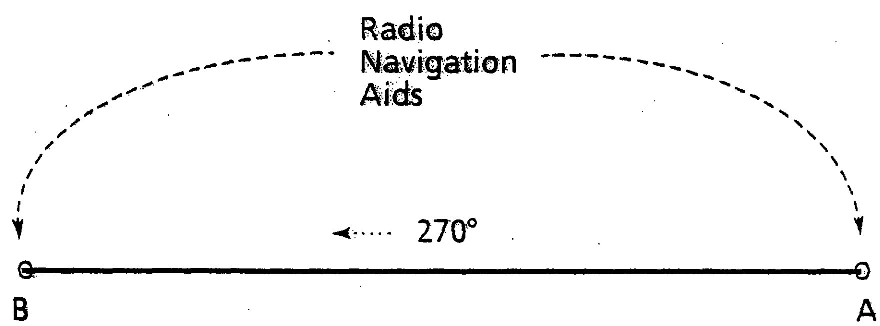

Figure 1-1
A Typical Airway

direction (here, 270° or due west), and is defined by two radio navigation aids: one at either end. A radio navigation aid is a radio transmitter that sends out a signal which can be received and interpreted by properly equipped aircraft, showing the aircraft's bearing from the navigation aid (that is, the direction <u>from</u> the nav aid <u>to</u> the aircraft) as well as, in many cases, the aircraft's distance from the nav aid. With this information the aircraft can fly along the airway in Figure 1-1 by maneuvering so as to maintain a bearing of 270 degrees from navaid A, or alternatively, a bearing of 090 degrees from nav aid B. A complex network, constructed by linking together simple airways like the one in Figure 1-1, is maintained by the FAA to encourage an orderly flow of both VFR and IFR traffic. The reader should note that IFR traffic does not always fly on airways - they sometimes fly "point-to-point," or directly from one location to another without regard to published airways, if they wish to do so and the FAA approves. Similarly, VFR aircraft often use published airways, particularly if they are flying long distances over unfamiliar terrain.

IFR and VFR traffic can also be distinguished by the altitudes they typically fly. VFR aircraft are often unpressurized, propeller-driven aircraft. Such aircraft fly more efficiently at relatively low altitudes (usually below 10,000 feet); further, their crew and passengers must breathe ambient air - air as it is found at the altitude they are flying - unless the aircraft is equipped with expensive personal oxygen systems. Since breathing ambient air above 10,000 feet or so can lead to mental confusion, disorientation and ultimately unconsciousness, we again find VFR traffic typically flying at altitudes below 10,000 feet. Finally, FAA regulations prohibit VFR flight above 18,000 feet.

IFR aircraft are typically jet powered and well equipped to fly in the more physiologically demanding altitudes above 18,000 feet. Further, jet aircraft tend to cruise most efficiently in the 25,000 - 45,000 feet altitude range, so they usually cruise well above the altitudes used by VFR traffic. Of course, both VFR and IFR traffic are found close to the ground in the vicinity of airports as they transition to or from the takeoff/landing phase of flight.

In order to monitor the safe and efficient flow of air traffic, the airspace above the United States is divided into twenty major parts, each of which is assigned to an FAA Air Route Traffic Control Center (ARTCC). Each of these "centers" is further divided into 30 or more sectors, with each sector in turn under the control of a specific air traffic controller. The controller in each sector provides a multitude of services to the aircraft flying through that sector, including weather and traffic advisories, and separtion monitoring. The controller provides these services through the use of a complex system of communication links between himself, the aircraft assigned to his sector, controllers in neighboring sectors, local weather forcasting facilities, and so on. In addition, the sector controller uses a nationwide, computer-aided radar monitoring system which provides information to a radar display screen located at the controller's work station. This radar display screen depicts a view "from above" of the controllers assigned airspace; it has the capability to display the airways crossing the sector, the position of each aircraft flying through the sector, as well as each aircraft's call sign, velocity, and altitude. Using this radar display, the controller can monitor the flight of all aircraft within his sector, thereby insuring that they follow their assigned routing. In addition, the controller can watch for possible violations of minimum separation standards. As the traffic flow becomes heavier, such violations become more likely, and controllers often find a good portion of their time dedicated to detecting such developing violations and issuing instructions to the aircraft involved to insure that minimum required separation is maintained.

The objective of this thesis is to develop an analytic model which will enable us to predict the rate at which sector controllers will be required to intervene in the flow of traffic to prevent violations of minimum separation standards. An ability to predict this intervention rate would be useful in several ways. First, it would enable FAA officials to quantify both the rate at which potential violations occur in the present air route traffic network, as well as the controller effort required to prevent such violations, near-misses, or perhaps even collisions. This ability would also permit an objective investigation into particular network configurations and traffic distributions which controllers claim result in an unacceptably high intervention rate.

A second benefit derived from the analytic model developed in this thesis would be an ability to predict intervention rates for sector geometries, network structures, and traffic flows that do not currently exist. This would allow a less subjective analysis of proposed changes to the current system, showing how intervention rates may change as a function of hypothetical changes in sector geometries, airway structures, traffic distributions and minimum separation standards. This kind of analytic capability will be of great benefit when considering the effects on controller intervention rates of future demands on the en route traffic system, as, for example, when reduced minimum separation standards are considered as a means to increase system capacity.

An analytic tool like the one proposed for this thesis does not currently exist in the literature. Models currently available have two significant limitations: they are limited to specific parts of the overall network (eg. a single intersection) without any indication of how they can be generalized to the more complex networks typical of actual en route sectors, and they model conflict rates rather than controller intervention rates.

A conflict rate is a measure of the rate at which conflicts can be expected to occur, where a conflict is defined to be a violation of the minimum acceptable separation between two airborne aircraft. Depending upon the magnitude of the minimum acceptable separation, a conflict may be a near-miss or (if the minimum separation is zero) an actual collision. In an uncontrolled environment, that is, one in which aircraft separation is not monitored by a radar-assisted air traffic controller, a model that accurately predicts conflict rates will provide a measure of the collision hazard in that particular area.

When aircraft operate under radar control, however, it may no longer be enough to ask how many conflicts would occur if no controller were present. A better question would be to inquire how often the controller can be expected to need to intervene in the regular flow of traffic by issuing instructions designed to avoid a potential conflict. This intervention rate will be a partial measure of controller workload, and thus allow observers to estimate the controllers' ability to insure proper separation between aircraft in his sector.

To see why conflict and intervention rates may not be the same, consider the situation depicted in Figure 1-2. Here we show a simple intersection of two airways, with aircraft A

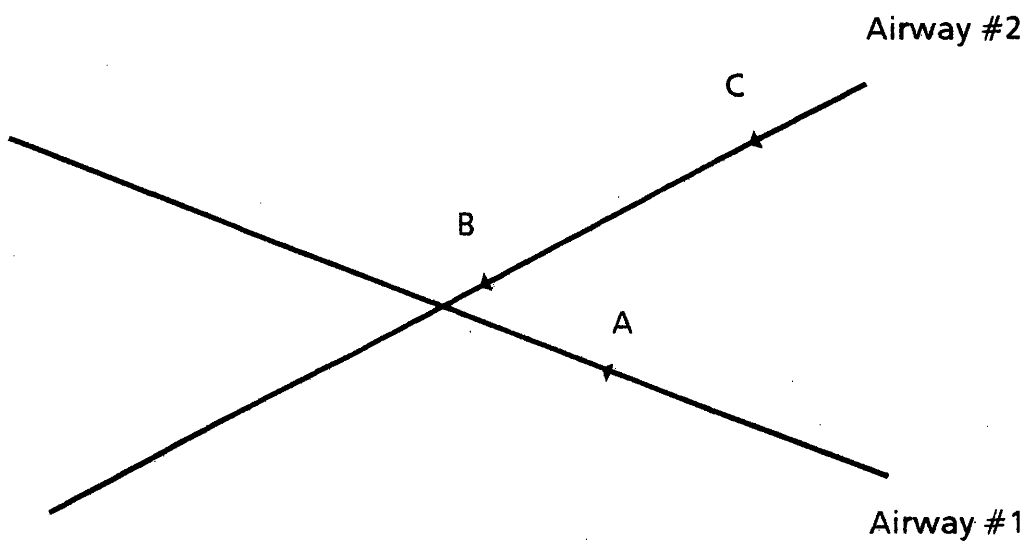

Figure 1-2

traveling along Airway #1 from lower right to upper left, and aircraft B and C traversing Airway #2 from upper right to lower left. None of the aircraft indend to change airways at the intersection. Now assume that aircraft B and C are traveling at the same velocity and are maintaining somewhat more than the minimum required separation between them, but aircraft A will conflict with both B and C as A passes the intersection. Thus we have two conflicts (one between A and B, and one between A and C). But, do we have two controller interventions?

A model which equates conflicts with interventions takes a very mechanistic view of controllers' reactions to impending conflicts. Controllers observed by this author under actual working conditions emphasized that there is no standard response to any developing conflict situation. Each has characteristics all its own, and each requires a somewhat different response from the controller. In the situation depicted in Figure 1-2, it would be unlikely that a controller would issue separate deviation instructions to both aircraft B and C. Instead, he would probably order a deviation of aircraft A around both B and C, thereby negating two potential conflicts with a single intervention. The model developed in this thesis will be designed to predict the rate at which controller interventions, not simply conflicts, can be expected to occur as a function of network structure, minimum separation requirements, and traffic characteristics (density, velocity).

We begin in Chapter 2 by constructing an analytic model which predicts controller intervention rates caused by both crossing and overtaking conflicts in a generalized two-dimensional (i.e. single altitude) en route sector. In Chapter 3 we will confirm the results of this model through Monte Carlo simulation, and also highlight the variance about the mean intervention rates predicted in Chapter 2 that can be expected in an operational en route sector. Chapter 4 will discuss the sensitivity of the analytic model to several key assumptions, including those involving interarrival distributions, cross-track deviations, and transient system parameters. Chapter 5 will extend the two-dimensional model to three dimensions (in the en route portion of the network) and show why the model as it stands will not extend to Terminal Control Areas. In Chapter 6 we will summarize, and present some possible opportunities for further research. In the Appendix we discuss some details on the delayed negative exponential distribution, which plays a central role in the model in Chapter 2. We will also examine some peripheral problems involving the use of geometrical probability in the modeling of aircraft collisions, and list the computer programs referred to in the body of the thesis.

1.2 A Review of the Literature

The best known of the early attempts to model a portion of the en route air traffic system is the Reich model (Reich [1966]), which was designed to predict collision rates in the trans-oceanic airspace over the North Atlantic. Reich used several approaches to the problem that have since been adopted and adapted by numerous authors. First, Reich's model counted conflicts (actually collisions) and not interventions - which made sense, since oceanic traffic was, and is , not radar-controlled. Secondly, Reich used a geometrical technique to predict collision rates by defining a conflict to be the violation of a prescribed "protected volume" about one aircraft by the center of gravity of another. By setting the dimensions of the protected volume to twice the dimensions of the typical airliner, each conflict represented a mid-air collision. Reich assumed that the probabilities of cross-track, along-track, and vertical conflicts were independent of one another, so his mathematical model of the collision rate took the form

$$R = R_X P_{Y|X} P_{Z|X} + R_Y P_{X|Y} P_{Z|Y} + R_Z P_{X|Z} P_{Y|Z} , \qquad (1-1)$$

where

R_X ≡ the rate at which cross-track overlaps occur

R_Y ≡ the rate at which along-track overlaps occur

R_Z ≡ the rate at which vertical overlaps occur

$P_{X|i}$ ≡ the conditional probability that cross-track overlap will occur, given that along-track (i = Y) or vertical (i = Z) overlap has occurred

$P_{Y|i}$ ≡ the conditional probability that along-track overlap will occur, given that cross-track (i = X) or vertical (i = Z) overlap has occurred

$P_{Z|i}$ ≡ the conditional probability that vertical overlap will occur, given that cross-track (i = X) or along-track (i = Y) overlap has occurred.

The Reich model has limited direct applicability to continental en route air traffic, since oceanic tracks are parallel and thus lack the numerous intersections which are characteristic of overland air traffic networks.

A model commonly applied to continental traffic is the so-called gas model (Flanagan [1962], Graham [1969], Endoh [1982]). This model derives its name from the fact that it uses the same protected volume technique used by Reich, while representing the centers of gravity of other aircraft as if they were gas molecules uniformly distributed throughout the airspace in question. The gas model has been developed in two- and three- dimensional versions. In either case, the key concept is that conflicts occur at a rate proportional to the volume generated by the protected volume as it moves through space. The actual conflict rate is the product of this generated volume times the density of the "gas" measured in aircraft centers of gravity per unit volume. One example of the mathematical formulation of a two-dimensional gas model (taken from Endoh [1092]) looks like

$$E = \frac{1}{2}\frac{N^2(2g)E(V_r)}{A} \qquad (1-2)$$

where

E $\equiv$ conflict rate

N $\equiv$ number of aircraft in the total airspace

A $\equiv$ the area of the total airspace

g $\equiv$ the horizontal dimension of the aircraft

$E(V_r)$ $\equiv$ the expected relative velocity between two aircraft in the airspace

The reason for the 1/2 multiplier is to avoid counting each conflict twice. Notice that

$$2g\,E(V_r)$$

is the area swept out by the protected airspace. while

$$\frac{N^2}{A} = N\left(\frac{N}{A}\right)$$

is the number of aircraft in the area times the traffic density; thus, the product gives the total collision rate.

The gas model is designed to predict conflict (or collision) rates in uncontrolled airspace. In controlled airspace it will still predict conflict rates well, by may overestimate controller intervention rates for the reasons discussed earlier.

In Dunlay [1972] we find a model which introduces some of the features incorporated in the basic model presented in this thesis. Dunlay's objective is to "describe a method for estimating the expected number of aircraft interactions that require controller intervention . . . " (Dunlay [1972], p. 2); he assumes that each potential conflict will require a controller intervention. Dunlay looks at both overtaking and crossing conflicts, and uses a delayed negative exponential distribution on interarrival times to reflect the effect of controller attempts to insure minimum separation. He also models the intersection of two airways which change direction at their intersection, but includes only two of the three cases we discuss in Chapter 2. Dunlays later work (Dunlay [1973], [1974], [1975]) incorporates the effects of human factors in the intervention process, including controller projections of miss distances as conflict situations develop.

Dunlay's work provides a foundation for many of the results derived in this thesis. In particular, his use of the delayed negative exponential interarrival distribution, and his approach to conflict rates at intersections where airways change ground track, were of major significance. This thesis expands upon Dunlay's work by introducing the concept of intervention rate, generalizing the treatment of individual intersections and airways to a complex two-dimensional sector, and then showing how the two-dimensional techniques can be further extended to three dimensions. In addition, we show the degree to which these results are sensitive to several key assumptions.

Endoh [1982] takes a different approach to the modeling of conflicts at a simple airway intersection by applying some of the techniques used in the gas model. He begins by considering the intersection of two airways as shown in Figure 1-3. He assumes that aircraft travel parallel to their assigned airway and are distributed uniformly to the left and right of their airway within a distance L of the centerline. For a two-dimensional intersection (i.e. all aircraft are assumed to be at the same altitude), the expected number of aircraft on airway #2 that an aircraft on airway #1 conflicts with per unit time (while the airway #1 aircraft is in the conflict area) is

$$\frac{gE(V_r)}{B_2 L} , \qquad (1-3)$$

where

$2g \equiv$ diameter of the protected area around an aircraft (assumed to be circular)

$B_2 \equiv$ expected along-track separation between successive aircraft on Airway #2

$E(V_r) \equiv$ expected relative velocity between aircraft on airway #1 and those on Airway #2.

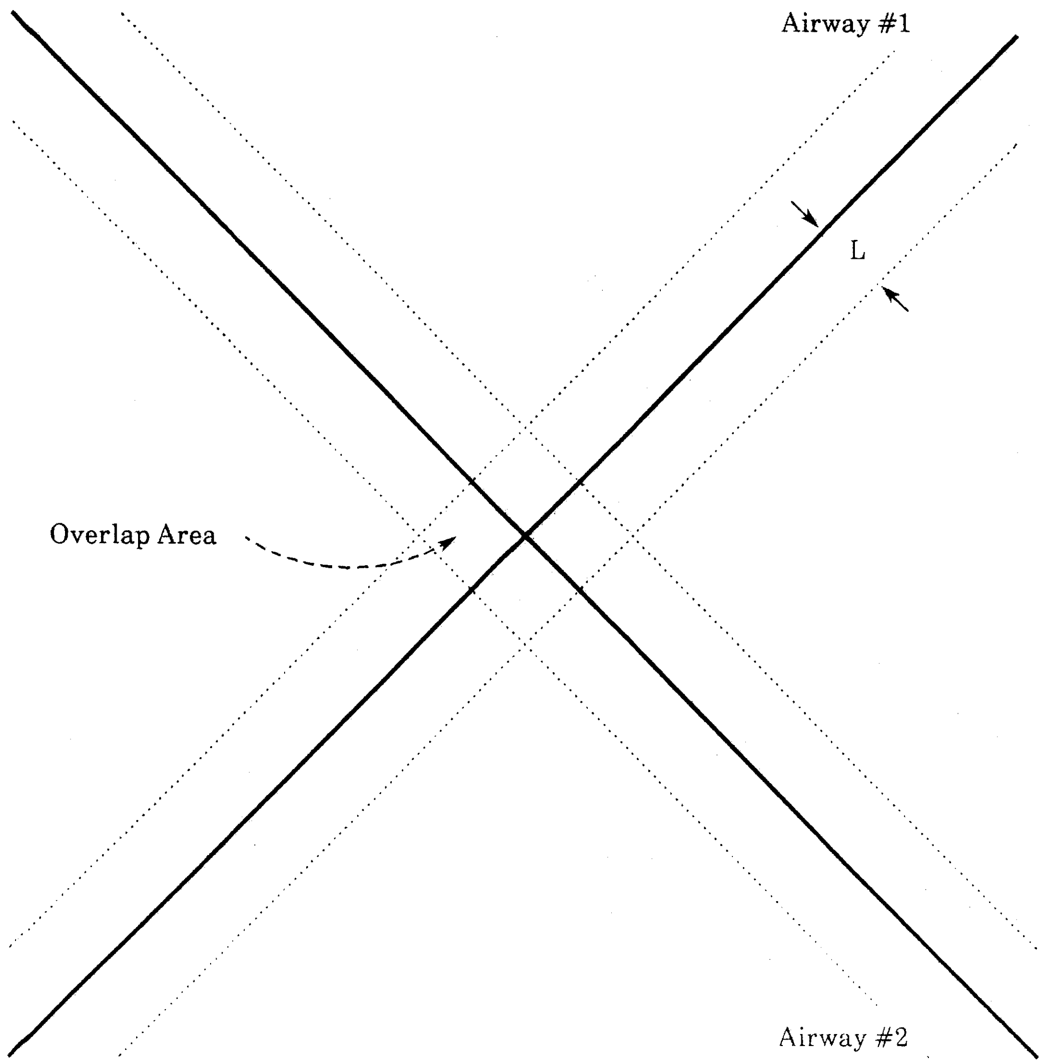

Figure 1-3

If the angle between the airways is α, and the aircraft on Airways #1 and #2 fly at velocities v_1 and v_2 respectively, then

$$E(V_r) = (v_1^2 + v_2^2 - 2v_1v_2\cos\alpha)^{\frac{1}{2}} .$$

Finally, since the expected number of aircraft on Airway #1 in the conflict area at any one time is

$$\frac{L}{B_1 \sin\alpha} ,$$

we get an intersection conflict rate

$$E = \frac{2g\,(v_1^2 + v_2^2 - 2v_1v_2\cos\alpha)^{\frac{1}{2}}}{B_1B_2\sin\alpha} . \qquad (1-4)$$

Note that this derivation assumes that L is large when compared with g, so that boundary effects can be ignored. This expression for E derived by Endoh confirms the earlier work in Dunlay [1972]. Dunlay's expression for the conflict rate at an intersection can be derived as a special case of Endoh [1982], equation (3-61).

Another novel approach to the problem of predicting conflict rates between aircraft traveling on airways passing close to one another can be found in Geisinger [1985]. This author generalizes to three dimensions in the sense that he does not restrict his airways to a constant altitude, although they are required to define a straight line in 3-space. Further, he considers airways which do not intersect but merely pass close to one another at some point. This approach allows Geisinger to apply his model to climb / descent paths in the vicinity of Terminal Control Areas. The two-dimensional airway intersection treated by Dunlay and Endoh, among others, is then simply a special case of Geisinger's more general formulation.

Geisinger assumes that aircraft always maintain centerline on their assigned airway, and that their (constant) velocity is known. Further, he assumes that the position of each aircraft along a given airway is uniformly distributed along an airway segment equal in length to the mean separation on the airway. These segments (each containing exactly one aircraft) follow one after the other along the airway, without overlaps or gaps between them.

1.3 Summary

In general, the literature contains numerous examples of attempts to model parts of the air traffic control system. Some authors treat highly specialized types of networks (eg. Reich's model), while others look at more general formulations of specific parts of a typical air traffic network, usually single intersections. Each author makes certain assumptions about aircraft velocities, interarrival distributions, and cross-track deviations, to name just a few. As a result, each highlights certain aspects of the problem, but none has proposed a general model with applicability to the complex structures characteristic of the modern en route air traffic system.

Such a general model is the objective of this thesis. We will construct an analytic model which will predict controller intervention rates due to both crossing and overtaking conflicts in a two-dimensional air traffic control sector to include various velocity distributions, aircraft ground-track changes at intersections, and off-airway traffic. We will also discuss the variation to be expected around predicted mean intervention rates with both analytic and Monte Carlo models. We will then show how the model may be sensitive to some of its key assumptions, and finally generalize the model to three dimensions, excluding Terminal Control Areas.

Chapter 2
A General Two-Dimensional Model of Controller Interventions in the En Route Air Traffic Structure

2.1 Controller Interventions Due to Crossing Conflicts

2.1.1 The Basic Model: The Intersection of Two Airways

To begin we are going to consider the following two-dimensional situation :

1. Airways #1 and #2 intersect at angle α ($0 \leq \alpha \leq 180°$) .

2. Aircraft travel on airway #1 at a constant velocity v_1 ; aircraft on airway #2 travel at a constant velocity v_2.

3. On airway i the expected separation between adjacent aircraft is S_i for $i \in \{1,2\}$.

4. The minimum acceptable separation between any two aircraft is a constant $M > 0$.

We wish to determine the rate at which controller interventions are required to avoid conflicts, where a conflict is defined to be a situation where two aircraft violate the minimum separation standard. We will assume that at most one intervention per intersection passage will be needed to avoid conflicts. In other words, when a single aircraft A is going to conflict

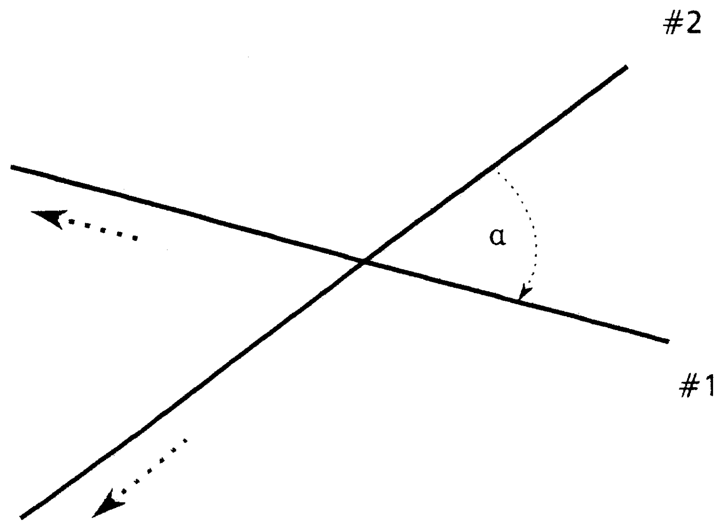

Figure 2-1

A Basic Airway Intersection

with two or more aircraft during intersection passage, one controller action (e.g. a vector or altitude change) will result in A's passage being conflict free.

One of the first questions to be addressed concerns the distribution of the random variable d, where d represents the distance between successive aircraft along a given airway. Remember we assume that $E[d] = S_i$ on airway i.

In Endoh [1982] the assumption is made that $d \sim NE(1/S_i)$; that is, the probability density function (pdf) for d is

$$f_d(x) = \begin{cases} \frac{1}{S_i} exp\left(-\frac{x}{S_i}\right) & 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases} \qquad (2-1)$$

This distribution is appealing due to its computational tractability, but it has the disadvantage of implying that two adjacent aircraft on either airway have some non-zero probability of being separated by a distance less than M. Clearly this shouldn't happen if adjacent sector controllers are doing their jobs. Remember, we are dealing with aircraft which travel at the same constant airspeed on any given airway.

Another approach, taken by Geisinger [1985], is to assume that aircraft are distributed along airway i, one to each segment of length S_i. Within each segment the assigned aircraft's position is distributed uniformly. This results in the following pdf (depicted graphically in Figure 2-2):

$$f_d(x) = \begin{cases} \dfrac{x}{S_i^2} & 0 \leq x \leq S_i \\[2ex] \dfrac{2}{S_i} - \dfrac{x}{S_i^2} & S_i \leq x \leq 2S_i \\[2ex] 0 & \text{otherwise} \end{cases} \qquad (2-2)$$

Again we have a distribution that is useful computationally but not terribly realistic for our PCA problem. It should be noted that both Geisinger's and Endoh's models may be more appropriate in other cases, where uncontrolled or highly controlled (sequenced) traffic is of interest.

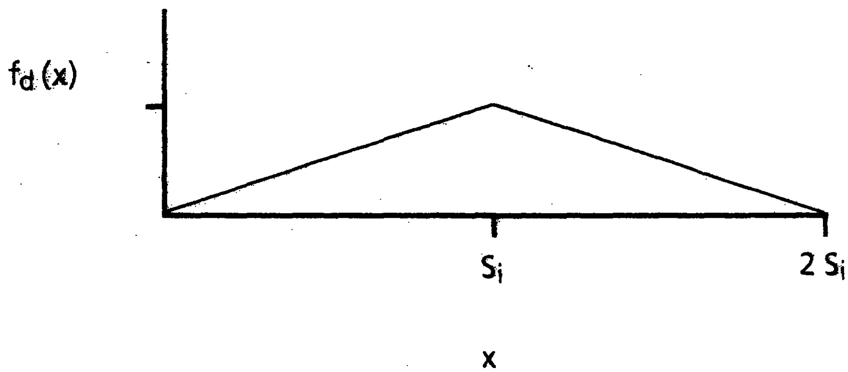

Figure 2-2

A Uniform Distribution within Segments of Length S_i

As an alternative to Endoh and Geisinger, we propose a "delayed negative exponential" (DNE) distribution on d according to:

$$f_d(x) \begin{cases} = 0 & x < M \\ = \dfrac{1}{S_i - M}\ exp\left(- \dfrac{x - M}{S_i - M} \right) & M \le x \end{cases} \qquad (2-3)$$

This distribution (see Figure 2-3) retains a reasonable computational tractability (note in particular the "conditional memoryless" feature : if $x \geq M$ then the distribution is memoryless) while maintaining $P[\, d < M \,] = 0$. It appears to be a more realistic model of the situation in question than either Endoh's or Geisinger's. Notice that $E[\, d \,] = S_i$.

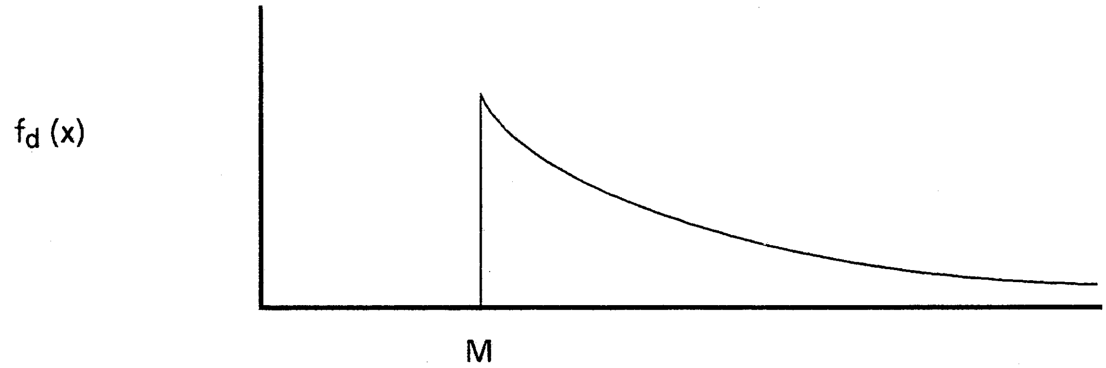

Figure 2-3

A Delayed Negative Exponential Distribution

<u>Theorem 1:</u> Given the assumptions associated with Figure 2-1. Let aircraft ① be an arbitrary aircraft on airway #1. Let aircraft ② be the nearest aircraft upstream on airway #2 at the instant ① crosses the intersection.

Let D ≡ the distance between ① and ② as ① crosses the intersection.

Then a conflict exists, or did exist, or will exist between ① and ② exactly when

$$D < MC$$

where

$$C \equiv [(KA)^2(1+k^2) + 1 + 2KA(\cos\alpha - k - KkA\cos\alpha)]^{-\frac{1}{2}} \qquad (2-4)$$

and

$$k \equiv \frac{v_2}{v_1}$$

$$A \equiv k - \cos \alpha$$

$$K \equiv (k^2 + 1 - 2k \cos \alpha)^{-1}$$

If $k=1$ and $\alpha=0$, then K is undefined; in this case $C \equiv 1$.

Remember, M is by definition the minimum allowable separation between aircraft.

<u>Proof:</u> Assume the hypothesis.

Let x be defined to be the distance between ① and the intersection; x is positive when ① is upstream from the intersection, and negative when ① has passed through the intersection.

By hypothesis, it follows that ② is a distance kx + D from the intersection on airway #2, with values greater than zero indicating ② is inbound to the intersection.

Let d be the Euclidean distance between ① and ②. Then by the Law of Cosines (see Figure 2-4),

$$d^2 = x^2 + (kx + D)^2 - 2x(kx + D)\cos \alpha$$

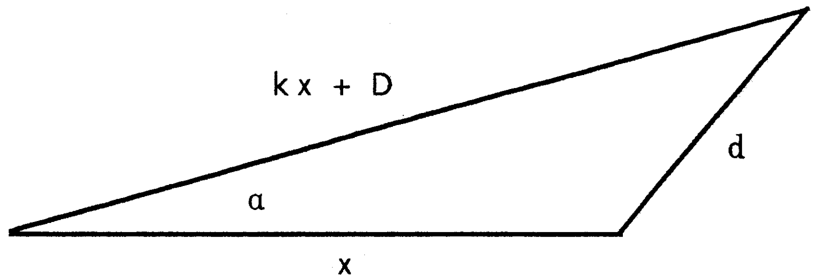

Figure 2-4

An Application of the Law of Cosines

Thus

$$d^2 = (k^2 + 1)x^2 + 2kxD + D^2 - 2kx^2\cos\alpha - 2xD\cos\alpha \tag{2-5}$$

Clearly $d^2 \to \infty$ as $x \to \pm\infty$, so d^2 is minimum when

$$\frac{d\,(d^2)}{dx} = 0.$$

But

$$\frac{d\,(d^2)}{dx} = 2(k^2 + 1)x + 2kD - 4kx\cos\alpha - 2D\cos\alpha$$

$$= x(2k^2 + 2 - 4\cos\alpha) + 2D(k - \cos\alpha)$$

so

$$\frac{d\,(d^2)}{dx} = 0$$

which implies

$$x = -\frac{2D(k - \cos \alpha)}{2k^2 + 2 - 4k\cos \alpha}$$

$$= -DAK \qquad (2-6)$$

We have, then, that the distance between ① and ② is minimum when $x = -DAK$; by the Law of Cosines this minimum distance is

$$S_M = [(-DAK)^2 + (-kDAK + D)^2 - 2(-DAK)(-kDAK + D)\cos \alpha]^{\frac{1}{2}} \qquad (2-7)$$

using $kx + D$ for the position of ②.

Since M is the minimum acceptable separation between aircraft, controller intervention will be required when

$$S_M < M$$

or

$$[(-DAK)^2 + (D - kDKA)^2 - 2(-DAK)(D - kDAK)\cos \alpha]^{\frac{1}{2}} < M$$

or

$$[D^2(AK)^2 + D^2 - D^2(2KkA) + D^2(KkA)^2 + D^2\{2KA(1 - KkA)\cos \alpha\}]^{\frac{1}{2}} < M$$

which gives

$$D[\ (KA)^2(1+k^2)\ +\ 1\ +\ 2KA(\cos\alpha - k - KkA\cos\alpha)]^{\frac{1}{2}}\ <\ M$$

or

$$D\ <\ MC, \qquad \text{as desired.}$$

In the unlikely event that $k=1$ and $\alpha=0$, K is not well defined. In this situation we can derive a value for C directly, since we have two airways superimposed on one another, with all aircraft traveling in the same direction at the same airspeed. Thus the distance between aircraft never change, and so a conflict between ① and ② occurs exactly when $D<M$. Therefore $C=1$ when $k=1$ and $\alpha=0$.

K is well defined in all other cases, and thus the desired result follows.

■

Now consider again the situation in Figure 2-1, with the interarrival distances on airway i distributed according to a delayed negative exponential distribution with mean S_i and delay M. Lets call the space of length M behind each aircraft which is empty of other aircraft a "null zone." Then we have

Lemma 2: Let ① be an arbitrary aircraft on airway #1, where airway #1 crosses airway #2 at angle α as in Figure 2-1. Then the probability that ① falls in a null zone on airway #2 as ① crosses the intersection of the two airways is

$$\frac{M}{S_2}.$$

Proof: This is a straight-forward random incidence proof, which runs as follows:

P [① is in a null zone given ① arrived in a gap (on airway #2) of length y]

$$= M / y \qquad \text{for } y \geq M$$

undefined otherwise

P [① arrived in a gap of length y]

$$= \frac{y f_d(y)}{E[y]}$$

Now E [y] = S_2 by hypothesis, and

$$f_d(y) \quad = \frac{1}{S_2 - M} \, exp\left(- \frac{y - M}{S_2 - M} \right) \qquad \textit{for } M \leq y$$

$$= 0 \qquad \textit{otherwise}$$

is our delayed negative exponential distribution.

Thus P [① arrived in a gap of length y]

$$= y\left(\frac{1}{S_2 - M}\right) exp\left(-\frac{y-M}{S_2 - M}\right)\frac{1}{S_2} \qquad for\ y \geq M$$

$$= \quad 0 \qquad otherwise$$

Consequently,

P [① arrives in a null zone and ① arrives in a gap of length y] =

P [① arrives in a null zone given ① arrives in a gap of length y]
· P [① arrives in a gap of length y]

$$= \frac{M}{y}\, y\left(\frac{1}{S_2 - M}\right) exp\left(-\frac{y-M}{S_2 - M}\right)\frac{1}{S_2} \qquad for\ y \geq M$$

$$= \quad 0 \qquad otherwise$$

and the total probability of ① arriving in a null zone is

$$\int_{-\infty}^{\infty} P\,[\text{ ① } arrives\ in\ a\ null\ zone\ and\ \text{ ① } arrives\ in\ a\ gap\ of\ length\ y\,]\ dy$$

$$= \int_{M}^{\infty} \left(\frac{M}{y}\right) y\left(\frac{1}{S_2 - M}\right) exp\left(-\frac{y-M}{S_2 - M}\right)\frac{1}{S_2}\ dy$$

$$= \frac{M}{S_2} \int_M^{\infty} \left(\frac{1}{S_2 - M} \right) exp\left(- \frac{y - M}{S_2 - M} \right) dy$$

$$= \frac{M}{S_2} \int_0^{\infty} \left(\frac{1}{S_2 - M} \right) exp\left(- \frac{z}{S_2 - M} \right) dz$$

$$= \frac{M}{S_2} \qquad\qquad \textit{as promised.}$$

■

By symmetry, we note that an arbitrary aircraft on airway #2 will be in a null zone as it crosses airway #1 with probability M / S_1.

Theorem 2: Given the situation as in Figure 2-1. Let ① be an arbitrary aircraft on airway #1. Let ② be the aircraft immediately upstream on airway #2 as ① crosses the intersection of the two airways. (See Figure 2-5). Then ① will pass within M of ② (i.e. a conflict between ① and ② will occur) withprobability

$$PCON_1 \equiv 1 - \left(\frac{S_2 - M}{S_2} \right) exp\left(\frac{M - CM}{S_2 - M} \right) \qquad (2-8)$$

where C is defined as in Theorem 1.

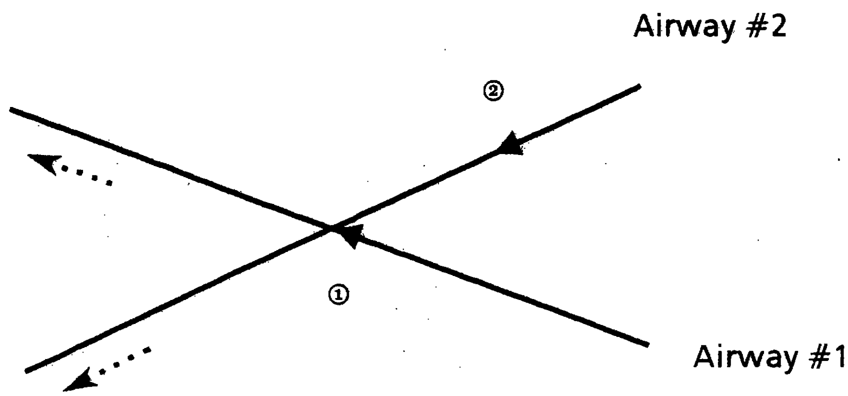

Figure 2-5

<u>Proof:</u> By Lemma 2, ① will be in an airway #2 null zone with probability M / S_2 ; ① will not be in a null zone with probability $1 - (M / S_2)$.

If ① is not in an airway #2 null zone as it crosses the intersection, then it is at least M behind ②'s predecessor on airway #2 (see Figure 2-6) . This puts ① in the memoryless portion of airway #2, which in turn means that the probability of a conflict with ② (i.e. ② is within CM of ①) is just

$$1 - exp\left(- \frac{CM}{S_2 - M} \right) . \qquad (2-9)$$

If ① <u>is</u> in an airway #2 null zone, figuring the probability of conflict with ② is a bit more complex.

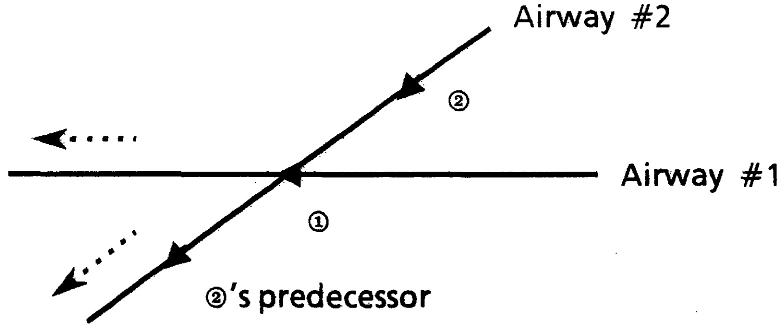

Figure 2-6

Consider the instant ① crosses the intersection. Let x be defined to be the distance from ① to ②'s predecessor on airway #2. It seems clear that x is uniformly distributed over the interval [0, M].

For any $x \in [0, M]$, the probability of ① conflicting with ② is simply the probability that ② is within CM of ①. Now ② is at least M - x away from ①, since ① is in a null zone. The probability that ① is within CM of ② is

$$1 - exp\left(- \frac{(CM + x) - M}{S_2 - M} \right) \qquad (2-10)$$

by the nature of our delayed negative exponential distribution.

Thus

P [① conflicts with ② given ① is within M of ②'s predecessor]

$$= \int_0^M P\,[\,distance\ from\ ①\ to\ ②\text{'s}\ predecessor = x\,]$$

$$\cdot\ P\,[\,①\ conflicts\ with\ ②\ given\ the\ distance\ from\ ①\ to\ ②\text{'s}\ pred. = x\,]\,dx$$

$$= \int_0^M \frac{1}{M}\left(1 - exp\left(-\frac{(CM + x) - M}{S_2 - M}\right)\right) dx$$

$$= \frac{1}{M}\left[x + (S_2 - M)\ exp\left(-\frac{(CM + x) - M}{S_2 - M}\right) \right]_0^M$$

$$= \frac{1}{M}\left[M + (S_2 - M)\ exp\left(-\frac{CM}{S_2 - M}\right) - (S_2 - M)\ exp\left(-\frac{CM - M}{S_2 - M}\right) \right]$$

$$= 1 + \frac{S_2 - M}{M}\left[exp\left(-\frac{CM}{S_2 - M}\right) - exp\left(-\frac{CM - M}{S_2 - M}\right)\right]$$

$$= 1 + \left(\frac{S_2 - M}{M}\right)\left(exp\left(-\frac{CM}{S_2 - M}\right)\right)\left(1 - exp\left(\frac{M}{S_2 - M}\right)\right) \qquad (2-11)$$

Combining (2-10) and (2-11) with Lemma 2 gives the total probability of conflict between ① and ② as

$$PCON_1 = \frac{M}{S_2}\left[1 + \left(\frac{S_2 - M}{M}\right)\left(exp\left(-\frac{CM}{S_2 - M}\right)\right)\left(1 - exp\left(\frac{M}{S_2 - M}\right)\right)\right]$$

$$+ \left(1 - \frac{M}{S_2}\right)\left(1 - exp\left(-\frac{CM}{S_2 - M}\right)\right)$$

$$= \quad 1 - \left(\frac{S_2 - M}{S_2}\right) exp\left(\frac{M - CM}{S_2 - M}\right)$$

as desired.

■

Once again, symmetry gives a similar expression for an arbitrary aircraft on airway #2 :

$$PCON_2 = 1 - \left(\frac{S_1 - M}{S_1}\right) exp\left(\frac{M - CM}{S_1 - M}\right) \qquad (2-12)$$

Corollary 2: Given the situation as in Figure 2-1.

The overall intervention rate at the intersection of airways 1 and 2 is

$$R_c \equiv \frac{v_1}{S_1} PCON_1 + \frac{v_2}{S_2} PCON_2 . \qquad (2-13)$$

$PCON_1$ and $PCON_2$ are as defined in Theorem 2.

Proof: For each $i \in \{1,2\}$, $PCON_i$ is the probability of conflict with an aircraft upstream on the other airway. Multiplying each $PCON_i$ by the appropriate traffic rate (i.e. v_i / S_i) and then summing the products will give the total intervention rate at the intersection.

Since $PCON_i$ considers only upstream conflicts, the total R_c counts each conflict exactly once.

$\blacksquare$

2.1.2 A Comparison of R_c vs. Endoh's Results

In Endoh [1982] we find an estimator of conflict rate at the intersection of two airways which, when restricted to two dimensions. looks like

$$E \equiv \frac{2M\left(v_1^2 + v_2^2 - 2v_1v_2\cos\alpha\right)^{\frac{1}{2}}}{S_1S_2\sin\alpha}, \qquad (2-14)$$

where M, v_1, v_2, S_1, S_2, and α are as we have defined them.

In Tables 2-1 through 2-5 we see R_c and E compared for various airspeeds, traffic densities, minimum separation distances M, and angles of intersection α.

α	M	v_1	v_2	S_1	S_2	E	R_c	$(E - R_c) / R_c$
10	.1	1	1	20	20	.00050	.00050	.0000
30	.1	1	1	20	20	.00052	.00052	.0000
45	.1	1	1	20	20	.00054	.00054	.0000
60	.1	1	1	20	20	.00058	.00058	.0000
90	.1	1	1	20	20	.00071	.00071	.0003
120	.1	1	1	20	20	.00100	.00100	.0012
135	.1	1	1	20	20	.00131	.00130	.0025
150	.1	1	1	20	20	.00193	.00192	.0053
170	.1	1	1	20	20	.00574	.00560	.0242
10	1	1	1	20	20	.00502	.00502	.0000
30	1	1	1	20	20	.00518	.00518	.0000
45	1	1	1	20	20	.00541	.00541	.0001
60	1	1	1	20	20	.00577	.00577	.0005
90	1	1	1	20	20	.00707	.00705	.0031
120	1	1	1	20	20	.01000	.00987	.0131
135	1	1	1	20	20	.01307	.01273	.0261
150	1	1	1	20	20	.01932	.01829	.0561
170	1	1	1	20	20	.05737	.04526	.2676

Table 2-1

α	M	v_1	v_2	S_1	S_2	E	R_c	$(E - R_c)/R_c$
10	5	1	1	20	20	.02510	.02510	.0000
30	5	1	1	20	20	.02588	.02588	.0002
45	5	1	1	20	20	.02706	.02703	.0010
60	5	1	1	20	20	.02887	.02877	.0034
90	5	1	1	20	20	.03536	.03467	.0197
120	5	1	1	20	20	.05000	.04626	.0808
135	5	1	1	20	20	.06533	.05619	.1625
150	5	1	1	20	20	.09659	.07113	.3580
170	5	1	1	20	20	.28684	.09772	1.9360
10	10	1	1	20	20	.05019	.05019	.0000
30	10	1	1	20	20	.05176	.05173	.0006
45	10	1	1	20	20	.05412	.05395	.0031
60	10	1	1	20	20	.05773	.05717	.0099
90	10	1	1	20	20	.07071	.06696	.0560
120	10	1	1	20	20	.01000	.08161	.2253
135	10	1	1	20	20	.13066	.09004	.4511
150	10	1	1	20	20	.19318	.09715	.9886
170	10	1	1	20	20	.57368	.10000	4.7370

Table 2-2

α	M	v_1	v_2	S_1	S_2	E	R_c	$(E - R_c) / R_c$
10	15	1	1	20	20	.07529	.07528	.0001
30	15	1	1	20	20	.07765	.07751	.0017
45	15	1	1	20	20	.08118	.08047	.0088
60	15	1	1	20	20	.08660	.08428	.0275
90	15	1	1	20	20	.10607	.09278	.1431
120	15	1	1	20	20	.14100	.09876	.5189
135	15	1	1	20	20	.19598	.09980	.9637
150	15	1	1	20	20	.28978	.10000	1.8980
170	15	1	1	20	20	.86052	.10000	7.6050
10	18	1	1	20	20	.09034	.09034	.0000
30	18	1	1	20	20	.09317	.09272	.0049
45	18	1	1	20	20	.09742	.09524	.0228
60	18	1	1	20	20	.10392	.09751	.0657
90	18	1	1	20	20	.12728	.09976	.2759
120	18	1	1	20	20	.17999	.10000	.8000
135	18	1	1	20	20	.23518	.10000	1.3520
150	18	1	1	20	20	.34773	.10000	2.4770
170	18	1	1	20	20	1.0326	.10000	9.3260

Table 2-3

α	M	v_1	v_2	S_1	S_2	E	R_c	$(E - R_c)/R_c$
30	10	1	3	20	20	.21918	.15186	.4430
60	10	1	3	20	20	.15275	.12316	.2400
90	10	1	3	20	20	.15811	.12607	.2540
120	10	1	3	20	20	.20816	.14805	.4060
150	10	1	3	20	20	.38982	.18481	1.1100
30	10	1	1	20	30	.03451	.03449	.0001
60	10	1	1	20	30	.03849	.03820	.0076
90	10	1	1	20	30	.04714	.04522	.0420
120	10	1	1	20	30	.06667	.05698	.1700
150	10	1	1	20	30	.12879	.07442	.7310
30	10	1	3	20	30	.14612	.11233	.3010
60	10	1	3	20	30	.10184	.08898	.1440
90	10	1	3	20	30	.10541	.09133	.1540
120	10	1	3	20	30	.13878	.10922	.2710
150	10	1	3	20	30	.25988	.13878	.8730
30	10	3	1	20	30	.14612	.10336	.4140
60	10	3	1	20	30	.10184	.08211	.2410
90	10	3	1	20	30	.10541	.08408	.2540
120	10	3	1	20	30	.13878	.10025	.3840
150	10	3	1	20	30	.25988	.13836	.8780

Table 2-4

From the data in Tables 2-1 through 2-4 the following observations can be made:

1. E and R_c are within a fraction of a percent of one another when the angle between the airways is small ($\alpha < 45°$), the traffic is light, and $v_1 \approx v_2$.

2. If $\alpha > 90°$, traffic density is relatively heavy, or if $v_1 \neq v_2$, then E and R_c can differ by as much as several hunderd percent.

3. At high α and traffic density, particularly when $v_1 \neq v_2$, E exceeds the traffic flow rate through the intersection by as much as 900% ; R_c approaches that rate asymptotically as $\alpha \rightarrow 180°$ and $M / S_i \rightarrow 1$.

Observations 1, 2, and 3 follow directly from the fact that E represents conflict rate while R_c measures controller intervention rate. In cases where multiple conflicts are unlikely (i.e. where an aircraft can be expected to conflict with at most one other aircraft) then we would expect to see a controller intervention for each potential conflict, and thus $E \approx R_c$. Intuitively, one would expect few multiple conflicts when α is small, the traffic density is light, and $v_1 \approx v_2$. This intuition is confirmed by observation 1.

Conversely, if α is near 180°, traffic density is high, and the velocities differ significantly, then one would expect a higher percentage of multiple conflicts. While E "counts" each of these conflicts separately, R_c does not. Remember, we assume that a single controller intervention will separate a given aircraft from all potential conflicts on the crossing airway. Thus E will tend to be significantly larger than R_c in these situations, as confirmed by observation 2.

Finally, in situations where nearly every aircraft crossing the intersection is in conflict, and when most conflicts are multiple, we can expect E to be larger than the flow rate through the intersection, since each aircraft will be expected to generate several conflicts. On the other hand, R_c will approach but not exceed the flow rate, since by definition at most one intervention per aircraft is required. Thus we have observation 3.

In Table 2-5 we see a comparison between E and R_c using values for S_i and v_i that are more typical of traffic actually observed in the high altitude structure.

α	M	S_1	S_2	v_1	v_2	E	R_c
30	5	60	60	360	360	1.0352	1.0352
60	5	60	60	360	360	1.1536	1.1536
90	5	60	60	360	360	1.4065	1.4065
120	5	60	60	360	360	1.9559	1.9558
150	5	60	60	360	360	3.5212	3.5212
30	5	60	60	300	540	1.7417	1.7417
60	5	60	60	300	540	1.4911	1.4911
90	5	60	60	300	540	1.6947	1.6947
120	5	60	60	300	540	2.2932	2.2932
150	5	60	60	300	540	4.0649	4.0649
30	5	60	40	300	540	2.5937	2.5936
60	5	60	40	300	540	2.2269	2.2268
90	5	60	40	300	540	2.5252	2.5252
120	.5	60	40	300	540	3.3922	3.3922
150	5	60	40	300	540	5.8702	5.8702

Table 2-5

2.1.3 Controller Intervention Rate at the Intersection of Two Airways Which Change Direction at the Intersection

In this section we will generalize the model to include airways which alter ground track at the intersection. We will continue to assume that the aircraft do not change airways at the intersection.

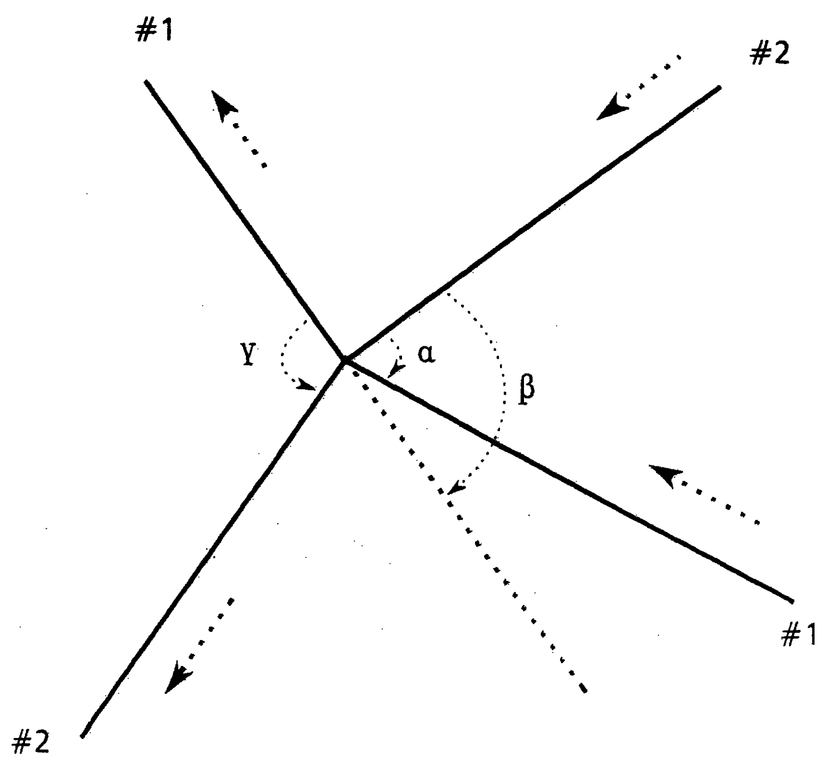

Figure 2-7

Two Airways Which Change Direction at Their Intersection

Let airways #1 and #2 intersect as shown in Figure 2-7. Define the following:

M = the minimum acceptable separation between aircraft

S_1 = the expected separation on airway #1

S_2 = the expected separation on airway #2

v_1 = the (constant) velocity of all aircraft on airway #1

v_2 = the (constant) velocity of all aircraft on airway #2

α = the angle between the inbound legs of the airways

β = the angle between the projection of the outbound leg of airway #1 back through the intersection, with the inbound leg of airway #2

γ = the angle between the outbound legs of the two airways

Assume that on both airways the distance between successive aircraft on the same airway is distributed according to a delayed negative exponential distribution with delay M and mean S_i for $i \in \{1,2\}$.

Consider an arbitrary aircraft ① on airway #1. Let ② be the closest aircraft to the the intersection that is inbound on airway #2 at the instant ① crosses the intersection. Finally, let D be the distance between ① and ② at the moment ① crosses the intersection. See Figure 2-8.

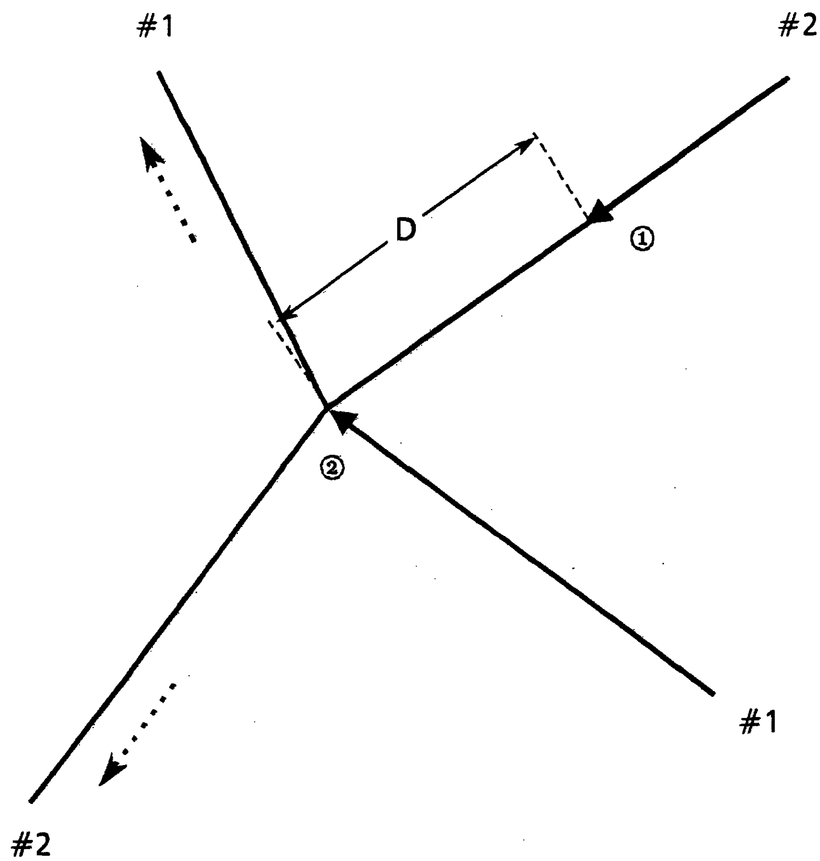

Figure 2-8

We want to determine how close ① and ② will pass as a function of D. To do this, we need to look at three cases, find the minimum separation in each case, and then pick the smallest of the three to determine the over-all minimum.

<u>Case 1:</u> Both ① and ② are inbound.

Let airways #1′ and #2′ intersect at angle α between both their inbound legs and their outbound legs as in Figure 2-9.

Let airways $\#1'$ and $\#2'$ have the same minimum separation, airspeeds, and mean separations as airways $\#1$ and $\#2$, respectively. That is,

$$M' = M$$

$$v_1' = v_1$$

$$v_2' = v_2$$

$$S_1' = S_1$$

$$S_2' = S_2$$

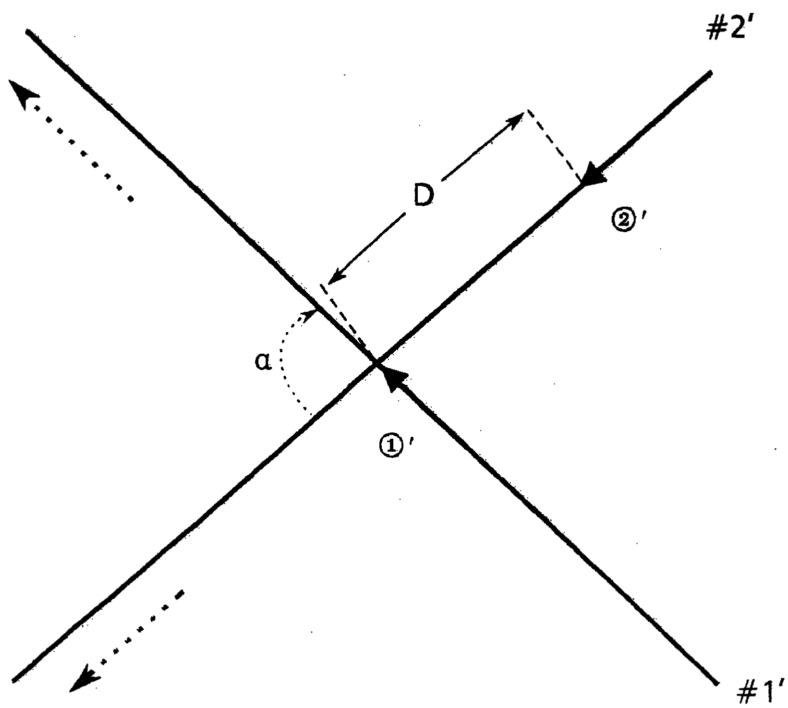

Figure 2-9

Case 1

Let ①′ be an arbitrary aircraft on airway #1′, and let ②′ be the first aircraft on airway #2′ upstream of ①′ as ①′ crosses the intersection. Define

x ≡ the distance between ①′ and the intersection
($x > 0$ when ①′ is inbound to the intersection)

By Theorem 1, the distance d′ between ①′ and ②′ is minimal when $x = -DAK$. Remember that

$$k \equiv \frac{v_2}{v_1}$$

$$A \equiv k - \cos\alpha$$

$$K \equiv (k^2 + 1 - 2k\cos\alpha)^{-1}$$

Notice that if k is small, A may be negative.

Notice also that as x decreases from $+\infty$, d′ decreases monotonically while $x > -DAK$, and increases monotonically when $x < -DAK$.

If $-DAK \geq 0$, then the minimum distance between ①′ and ②′ is achieved when both aircraft are still inbound to the intersection. In the proof of Theorem 1 we saw that this minimum distance was just

$$S_M \equiv [\,(-DAK)^2 + (D - kDAK)^2 - 2(-DAK)(D - kDAK)\cos\alpha\,]^{\frac{1}{2}}$$

by the Law of Cosines. If, on the other hand, $-DAK < 0$, then the minimum distance between ①′ and ②′ will be achieved after ①′ has passed through the intersection, so the minimum distance between ①′ and ②′ <u>while both are still inbound</u> is just D, since δ' is monotonically decreasing until x = −DAK.

Finally, the minimum distance between ①′ and ②′ <u>while both are still inbound</u> is exactly the minimum distance between ① and ② in Case 1; lets call this minimum distance δ^1. Then

$$\delta^1 \quad \begin{cases} = D & if \quad -DAK < 0 \\ = S_M & if \quad -DAK \geq 0 \end{cases} \qquad (2-15)$$

In the special case where $\alpha=0$ and $k=1$, K will be undefined. But this would mean that airways #1 and #2 were superimposed inbound to the intersection; thus the minimum distance between ① and ② while both were inbound to the intersection would be their constant separation D. Thus $\alpha=0$ and $k=1$ implies $\delta^1=D$.

<u>Case 2</u>: ① is outbound, ② is inbound.

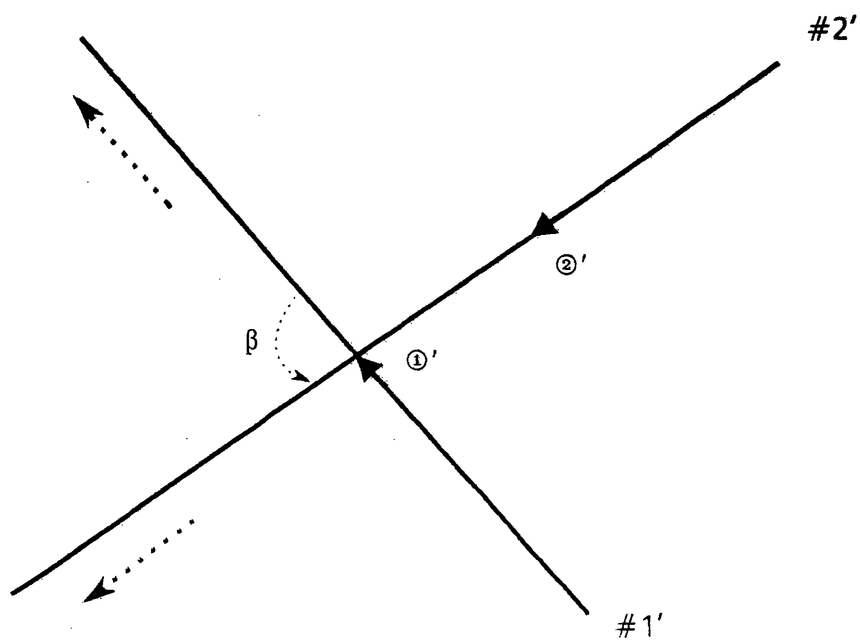

Figure 2-10

Case 2

Let airways #1′ and #2′ intersect at angle β, as in Figure 2-10. All other parameters (M', S_1', S_2', $v1'$, v_2') remain the same.

Let aircraft ①′ be at the intersection, with ②′ its closest neighbor upstream on airway #2′. Let D be the distance between them as ①′ crosses the intersection. Define x as before.

Once again, by Theorem 1 we have the minimum distance between ①′ and ②′ achieved when $x = -DAK$, where A and K are now functions of β. Notice that ①′ is outbound and ②′ is inbound exactly when

$$-\left(\frac{D}{v_2}\right)v_1 \le x \le 0$$

since (D/v_2) is the time between ①′ and ②′ crossing of the intersection.

If $0 < -DAK$, then the minimum distance between ①′ and ②′ <u>while ①′ is outbound and ②′ is inbound</u> is simply D.

If $-(D/v_2)\,v_1 \le -DAK \le 0$, then the minimum distance between ①′ and ②′ <u>while ①′ is outbound and ②′ is inbound</u> is S_M, defined as before.

If $-DAK < -(D/v_2)\,v_1$, then the minimum distance between ①′ and ②′ <u>while ①′ is outbound and ②′ is inbound</u> is $(D/v_2)\,v_1$.

These conclusions follow since the distance between ①′ and ②′ is monotonically decreasing while $x > -DAK$, and monotonically increasing thereafter, as $x \to -\infty$.

In the case where $\beta = 0$ and $k = 1$, K is undefined. But $\beta = 0$ means the outbound leg of airway #1 is parallel to the inbound leg of #2 (or, equivalently, #1′ and #2′ are superimposed). Since $k = 1$ means $v_1 = v_2$, it follows directly that the minimum distance between ①′ and ②′ will be equal to D.

Thus, since we chose the angle between airways #1′ and #2′ to be β, the minimum distance between ① and ② in Case 2 is

$$
\delta^2 = \begin{cases} D & \text{if } 0 < -DAK \\ & \text{or if } \beta = 0 \text{ and } k = 1 \\ S_M & \text{if } -\left(\dfrac{D}{v_2}\right)v_1 \leq -DAK \leq 0 \\ \left(\dfrac{D}{v_2}\right)v_1 & \text{if } -DAK \leq -\left(\dfrac{D}{v_2}\right)v_1 \end{cases} \tag{2-16}
$$

where A, K, and S_M are functions of β.

<u>Case 3</u>: Both ① and ② are outbound.

Using an argument similar to the one in the previous Cases, we consider airways #1′ and #2′ which intersect at angle γ. We find that

$$
\delta^3 = \begin{cases} \left(\dfrac{D}{v_2}\right)v_1 & \text{if } -\left(\dfrac{D}{v_2}\right)v_1 \leq -DAK \\ S_M & \text{otherwise} \end{cases} \tag{2-17}
$$

where A, K, and S_M are now functions of γ.

Once again, if $\gamma = 0$ and $k = 1$, then $\delta^3 = D$.

We have now proved the following:

Theorem 3: Let airways #1 and #2 intersect as shown in Figure 2-7, with M, v_1, v_2, S_1, and S_2 given as before. Let ① be an arbitrary aircraft on airway #1, and let ② be the closest inbound aircraft on airway #2 at the instant ① crosses the intersection, with D being the distance between them at that time.

Define

$$k \equiv \frac{v_2}{v_1}$$

$$A(\theta) \equiv k - \cos\theta$$

$$K(\theta) \equiv (k^2 + 1 - 2k\cos\theta)^{-1}$$

for arbitrary θ.

Then the minimum distance between ① and ② will be

$$\delta \equiv \min\{\delta^1, \delta^2, \delta^3\}$$

where

$$\delta^1 \begin{cases} \equiv D & \text{if } -DA(\alpha)K(\alpha) < 0 \\ & \text{or if } \alpha = 0 \text{ and } k = 1 \\ \equiv S_M(\alpha) & \text{otherwise} \end{cases}$$

$$\delta^2 \begin{cases} \equiv D & \text{if } 0 < -DA(\beta)K(\beta) \\ & \text{or if } \beta = 0 \text{ and } k = 1 \\ \equiv S_M(\beta) & \text{if } -\left(\frac{D}{v_2}\right)v_1 \leq -DA(\beta)K(\beta) \leq 0 \\ \equiv \left(\frac{D}{v_2}\right)v_1 & \text{otherwise} \end{cases}$$

$$\delta^3 \begin{cases} \equiv \left(\frac{D}{v_2}\right)v_1 & \text{if } -\left(\frac{D}{v_2}\right)v_1 \leq -DA(\gamma)K(\gamma) \\ & \text{or if } \gamma = 0 \text{ and } k = 1 \\ \equiv S_M(\gamma) & \text{otherwise} \end{cases}$$

and

$$S_M(\theta) \equiv [(-DA(\theta)K(\theta))^2 + (-kDA(\theta)K(\theta) + D)^2 - 2(-DA(\theta)K(\theta))(D - kDA(\theta)K(\theta))\cos\theta\,]^{\frac{1}{2}}$$

for arbitrary θ.

∎

By symmetry, we can also make a similar calculation for an arbitrary ② on airway #2 , with k redefined to be

$$k \equiv \frac{v_1}{v_2} .$$

<u>Corollary 3.1</u>: Assume the same hypothesis as in Theorem 3.

Then ① will pass within M of ② iff $D < MC$, where

$$C = \begin{cases} 1 & \text{if} \quad \left(\delta = \delta^1 \ \ and \ \ A(\alpha)K(\alpha) > 0 \right) \ or\,if \ \left(\delta = \delta^2 \ \ and \ \ 0 > A(\beta)K(\beta) \right) \\ \dfrac{v_2}{v_1} & \text{if} \quad \left(\delta = \delta^2 \ \ and \ \ A(\beta)K(\beta) \geq \left(\dfrac{v_1}{v_2} \right) \right) or\,if \left(\delta = \delta^3 \ \ and \ \ \left(\dfrac{v_1}{v_2} \right) \geq A(\gamma)K(\gamma) \right) \end{cases}$$

and otherwise

$$C = [\ (K(\theta)A(\theta))^2(1 + k^2) + 1 + 2K(\theta)A(\theta)(\cos\theta - k - KkA\cos\theta)\ ]^{-\frac{1}{2}} \tag{2-18}$$

$$\text{with } \theta = \begin{cases} \alpha & \text{if } \delta = \delta^1 \\ \beta & \text{if } \delta = \delta^2 \\ \gamma & \text{if } \delta = \delta^3 . \end{cases}$$

Proof: The five possible values for C correspond to the five possible values for δ in Theorem 3.

If $\delta = D$, then clearly $C = 1$.

If $\delta = (D/v_2)\, v_1$, then $C = v_2 / v_1$.

If $\delta = S_M(\theta)$ for $\theta \in \{\alpha,\beta,\gamma\}$, then by Theorem 1 we have C as in (4).

■

Once again, symmetry allows us to calculate C for an arbitrary ② on airway #2 , defining

$$k \equiv \frac{v_1}{v_2}$$

Theorem 4: Assume the same hypothesis as in Theorem 3.

Then ① will pass within M of ② (i.e. a conflict between ① and ② will occur) with probability

$$PCON = 1 - \left(\frac{S_2 - M}{S_2} \right) exp\left(\frac{M - CM}{S_2 - M} \right) \qquad (2-19)$$

where C is as in Corollary 3.1.

Proof: The proof of this theorem proceeds exactly along the lines of the proof of Theorem 2: the fact that the airways change ground track is of no significance.

■

Again, symmetry allows a similar theorem for an arbitrary ② on airway #2. To avoid confusion we make use of the following notation when necessary:

$$k_{i,j} \equiv \frac{v_j}{v_i}$$

$$A_{i,j}(\theta) \equiv k_{i,j} - \cos\theta$$

$$K_{i,j}(\theta) \equiv \left(k_{i,j}^2 + 1 - 2k_{i,j} \cos\theta \right)^{-1}$$

$C_{i,j} \equiv C$ for an aircraft on airway i, relative to airway j

$PCON_{i,j}$ ≡ the probability of conflict for an arbitrary aircraft on airway i awith an inboundircraft on airway j

Using this notation, we have

Corollary 4: Assume the hypothesis as in Theorem 3.

The overall intervention rate at the intersection of airways #1 and #2 is

$$R_c \equiv \frac{v_1}{S_1}(PCON_{1,2}) + \frac{v_2}{S_2}(PCON_{2,1}). \tag{2-20}$$

Proof: The total intervention rate is simply the sum of the probability of conflict on each airway times the flow rate along that airway. Since each $PCON_{i,j}$ "counts" only conflicts with upstream aircraft, no conflict is counted twice.

■

2.1.4 Controller Intervention Rate at the Intersection of N Airways

In previous sections we have developed a model which predicts controller intervention rates at the intersection of two airways, where

1. the minimum acceptable separation is $M > 0$
2. all aircraft on airway i travel at a constant speed v_i
3. airway i has traffic density $1/S_i$
4. interaircraft distances on a given airway are distributed according to a delayed negative exponential distribution with delay M and mean S_i.

In this section we will generalize to consider the intersection of an arbitrary N airways. Consider the situation as illustrated in Figure 2-11. Notice that the airways make (unspecified) ground track changes. For each ordered pair $(i,j) \varepsilon \{1,2,\ldots,N\} \times \{1,2,\ldots,N\}$ such that $i \neq j$, we measure

$\alpha_{i,j}$ $\equiv$ the angle between the inbound legs of airways i and j

$\beta_{i,j}$ $\equiv$ the angle between the extension back through the intersection of airway i's outbound leg, with the inbound leg of airway j

$\gamma_{i,j}$ $\equiv$ the angle between the outbound legs of airways i and j

as in Figure 2-12. Notice that $\alpha_{i,j} = \alpha_{j,i}$ and $\gamma_{i,j} = \gamma_{j,i}$; but in general $\beta_{i,j} \neq \beta_{j,i}$.

Consider an arbitrary aircraft A approaching the intersection in Figure 2-12 along airway i. For any $j \varepsilon \{1,2,\ldots,N\}$, $j \neq i$, we can conpute $PCON_{i,j}$ as in Theorem 4. Consequently, we can see that the probability that A does <u>not</u> conflict with any inbound aircraft on airway j is simply $1 - PCON_{i,j}$.

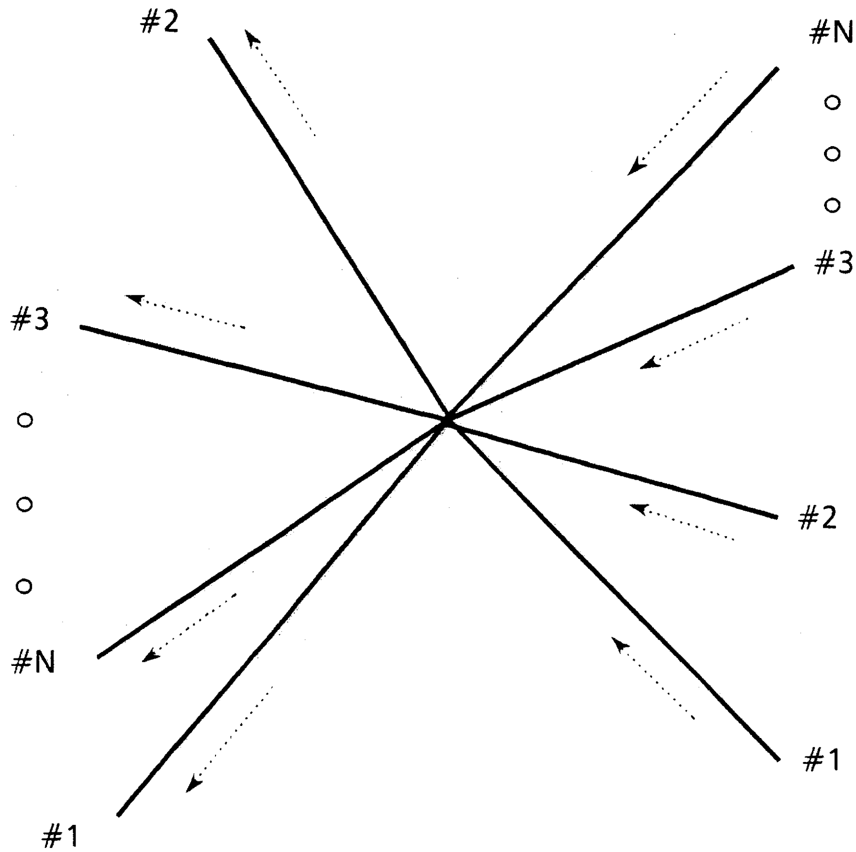

Figure 2-11

The Intersection of N Airways

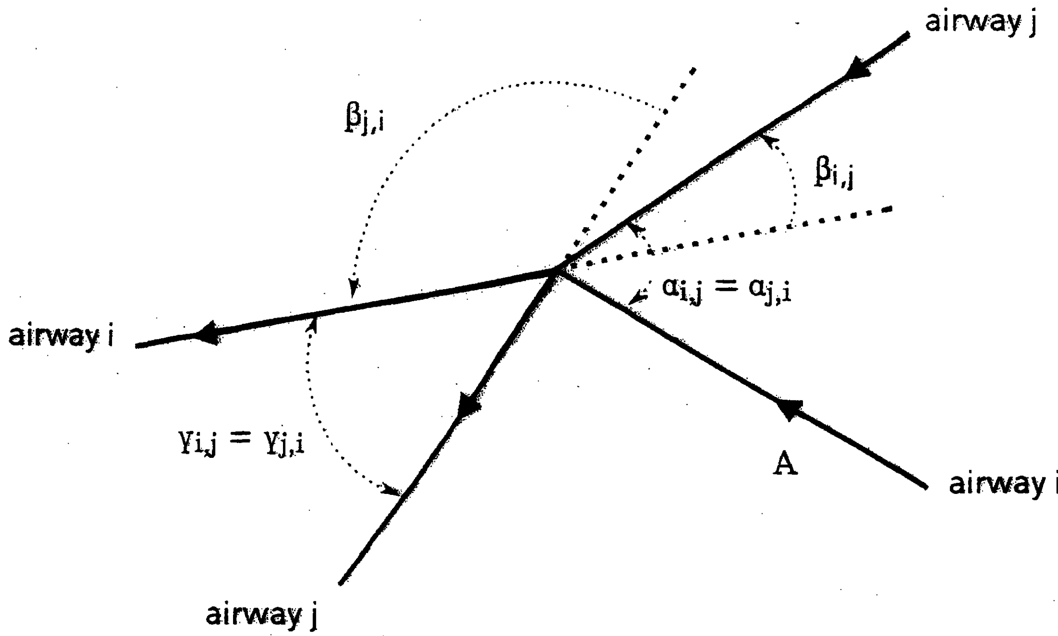

Figure 2-12

Since the flow on each airway is independent of the flow on any others, it follows that the probability that A passes through the intersection without conflicting with any inbound aircraft is simply

$$\prod_{\substack{j=1 \\ j \neq i}}^{N} (1 - PCON_{i,j}) ,$$

which means that the probability that A's passage of the intersection generates a controller intervention is just

$$PCON_i \equiv 1 - \prod_{\substack{j=1 \\ j \neq i}}^{N} (1 - PCON_{i,j}) . \qquad (2-21)$$

Remember, we are "counting" conflicts between A and aircraft on other airways that are still inbound to the intersection as A crosses it. Thus if we multiply each $PCON_i$ by the traffic density on airway i, the sum of these products over i will be the total controller intervention rate at the intersection. This gives us:

Theorem 5: Consider the intersection of N airways as shown in Figure 12, with $\alpha_{i,j}$, $\beta_{i,j}$, and $\gamma_{i,j}$ defined for each pair (i,j) of airways as shown in Figure 1-12.

Then the total controller intervention rate R_c at the intersection is

$$R_c = \sum_{i=1}^{N} \frac{v_i}{S_i} \left[1 - \prod_{\substack{j=1 \\ j \neq i}}^{N} (1 - PCON_{i,j}) \right]$$

$$= \sum_{i=1}^{N} \frac{v_i}{S_i} PCON_i \qquad (2-22)$$

■

2.1.5 Airway Changes at the Intersection

To incorporate the possibility that aircraft may change airways at the intersection we will use the technique of creating pseudo-airways, one for each possible combination of inbound (to the intersection) and outbound tracks.

Consider as an example the situation shown in Figure 2-13.

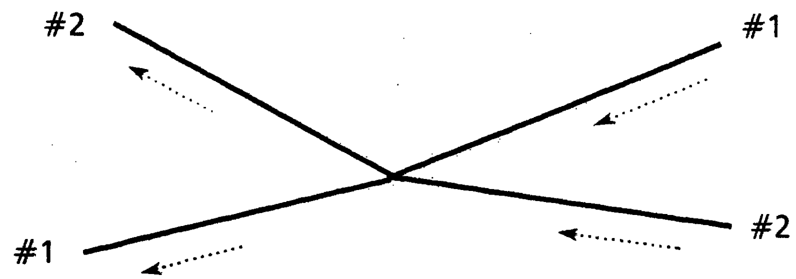

Figure 2-13

An Intersection Permitting Airway Crossovers

Let aircraft enter the two airways at rates λ_1 and λ_2 respectively, and assume that aircraft do not change airspeed during their flight through the sector .

Now suppose that 25% of the aircraft which approach the intersection on airway #1 switch to airway #2 at the intersection; similarly, 40% of the aircraft on airway #2 change to #1 at the intersection. Then we can compute R_c , the controller intervention rate due to crossing conflicts, by considering the four pseudo-airways shown in Table 2-6.

Pseudo-airway A, for example, has the same ground track as airway #1, but only 75% of #1's arrival rate. Pseudo-airway C , on the other hand, follows airway #2's ground track into the intersection, but then departs the intersection on airway #1's ground track. C has only 40% of the traffic density of airway #2. Note that aircraft on the same pseudo-airway fly the same (constant) airspeed.

Pseudo-Airway	Inbound Track	Outbound Track	Entry Rate
A	#1	#1	$(3/4)\lambda_1$
B	#1	#2	$(1/4)\lambda_1$
C	#2	#1	$(4/10)\lambda_2$
D	#2	#2	$(6/10)\lambda_2$

Table 2-6
Pseudo-Airways Applied to the Situation in Figure 2-13

First we should notice that relative to any single pseudo-airway X, the remaining pseudo-airways fall into exactly one of three classes:

I Those that share an inbound leg with X

II Those that share an outbound leg with X

III Those that meet X only at the intersection

If we pick pseudo-airway A from the example in Table 2-6, the three classes are {B}, {C}, and {D} respectively.

Now if we want to calculate the probability that an arbitrary aircraft on pseudo-airway X will conflict with an aircraft upstream on pseudo-airway Y, we find that the calculation will follow one of our previous theorems, depending on Y's class relative to X. Remember, aircraft ② is "upstream" of aircraft ① if it has not yet reached the intersection at the moment that ① does.

<u>CLASS I</u>:

If Y is in class I relative to X, then both aircraft (call them ① and ② in Figure 2-14) follow the same ground track into the intersection. Note that ② must be behind ①, since we

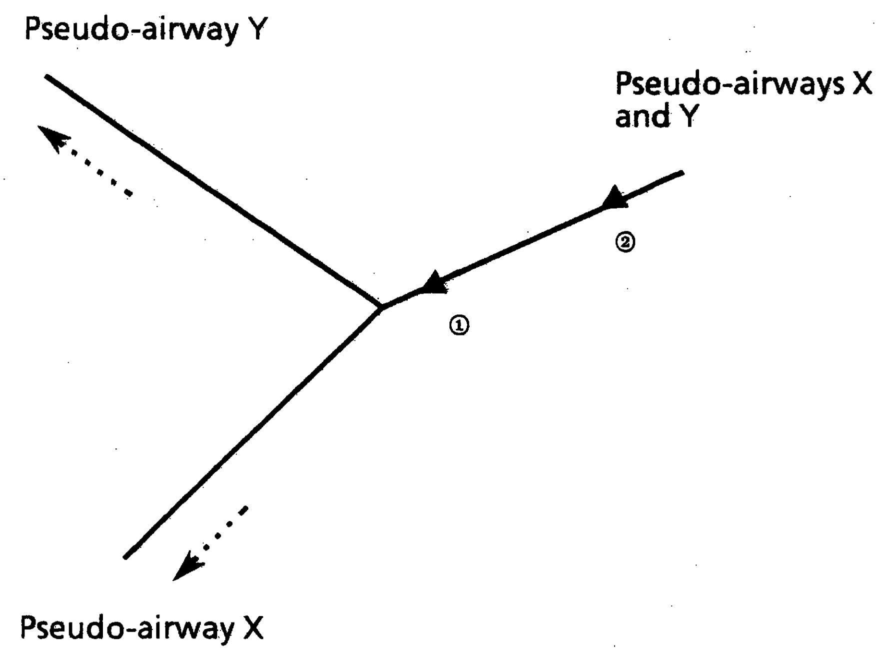

Figure 2-14

Two Pseudo-airways with a Common Inbound Track

are only counting conflicts between ① and those aircraft upstream from ①. If ① and ② conflict before ① reaches the intersection, then we have an overtaking situation, to be considered later. Thus, the only time ① and ② will generate a crossing conflict is after ① has passed through the intersection. This gives us two cases to consider, which we number to parallel the discussion leading up to Theorem 3:

Case 2 : ① is outbound and ② is inbound

Case 3: Both ① and ② are outbound

First lets adapt the following variables to this situation. The reader should notice the parallels to the argument in previous sections. In particular α, which was the angle between the inbound legs of the two airways, is now zero, so $\cos\alpha$ equals 1.

$D \equiv$ the distance between aircraft ① and ② as ① crosses the intersection

$v_1 \equiv$ the velocity of aircraft ①

$v_2 \equiv$ the velocity of aircraft ②

$k \equiv v_2/v_1$

$A(\theta) \equiv k - \cos\theta$

$K(\theta) \equiv (k^2 + 1 - 2k\cos\theta)^{-1}$

Note: if $k=1$ and $\theta=0$ (eg. $\theta=\alpha$), then $K(\theta)$ is undefined.

$\alpha \equiv$ the angle between the inbound legs of the pseudo-airways

$\beta \equiv$ the angle between the inbound leg of airway X (or Y) and an extension of X's outbound leg back through the intersection

$\gamma \equiv$ the angle between the outbound legs of X and Y

$$S_M(\theta) \equiv [\,(-DA(\theta)K(\theta))^2 + (D - kDA(\theta)K(\theta))^2 - 2(-DA(\theta)K(\theta))(D - kDA(\theta)K(\theta))\cos\theta\,]^{1/2}$$

Note: if $k=1$ and $\theta=0$, then $S_M(\theta)$ is undefined

The general thrust of the following argument parallels that of Theorem 3. We will first find the minimum distance between ① and ② as a function of D in cases 2 and 3. Taking the minimum of these minina, we will then compute the value for C such that a conflict occurs exactly when $D < MC$. Finally, we will calculate the probability that D will be less than MC.

Case 2: ① is outbound and ② is inbound.

The situation is as in Figure 2-15.

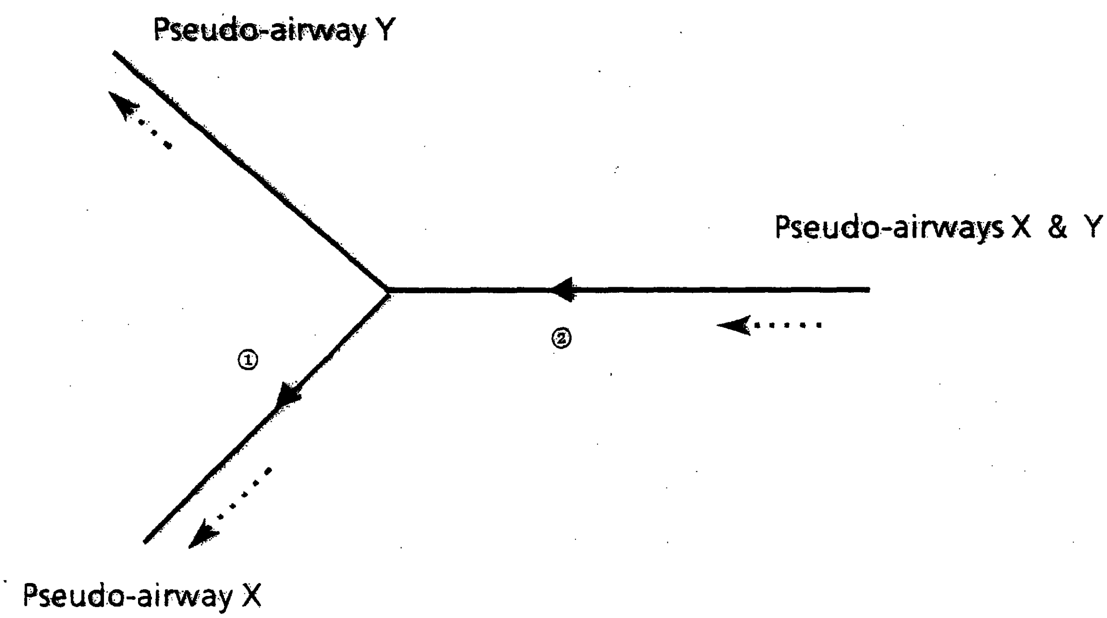

Figure 2-15

Two Pseudo-airways with a Common Inbound Track

If we define x to be ①'s distance from the intersection (with $x > 0$ when ① is inbound), then ②'s distance from the intersection as a function of x will be $kx + D$. The distance d between ① and ② in Case 2 can be determined once again using the Law of Cosines (Figure 1-16) to be

$$d = (x^2 + (kx + D)^2 - 2x(kx - D)\cos\beta)^{\frac{1}{2}} .$$

This follows since $\cos\beta = -\cos(\pi - \beta)$. We saw in 1.A.2 that the minimum value for d will occur when $x = -DA(\beta)K(\beta)$. Since d is monotonically increasing while $x > -DA(\beta)K(\beta)$ and monotonically decreasing when $x < -DA(\beta)K(\beta)$, we see once again that in Case 2,

$$
\min\{d\} = \begin{cases} D & \text{if } -DA(\beta)K(\beta) > 0 \\ & \text{or if } k=1 \text{ and } \beta=0 \\ S_M(\beta) & \text{if } -(D/v_2)\,v_1 \le -DA(\beta)K(\beta) \le 0 \\ (D/v_2)\,v_1 & \text{if } -DA(\beta)K(\beta) \le -(D/v_2)\,v_1 . \end{cases}
$$

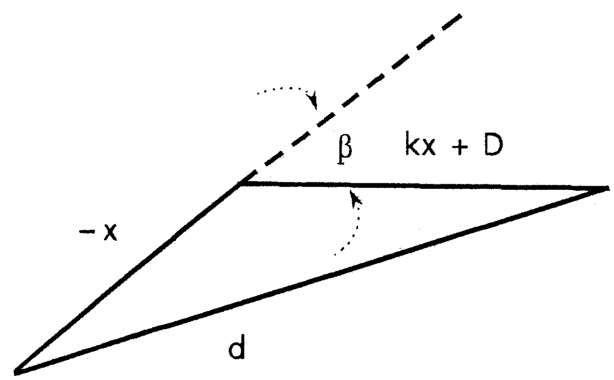

Figure 2-16

An Application of the Law of Cosines

<u>Case 3</u>: Both ① and ② are outbound.

A similar argument shows that the minimum distance between ① and ② is

$$
(D/v_2)\,v_1 \qquad \text{if } -(D/v_2)\,v_1 \le -DA(\gamma)K(\gamma)
$$
$$
\text{or if } k=1 \text{ and } \gamma=0,
$$

and

$S_M(\gamma)$ otherwise.

If we define δ^2 (where the 2 is a superscript, not an exponent) to be the minimum value of d in Case 2 and δ^3 to be the minimum value of d in Case 3, then the closest that ① and ② will pass together is

$$\delta \equiv \min\{\delta^2, \delta^3\}$$

where

$$\delta^2 \begin{cases} = D & \text{if} \quad -DA(\beta)K(\beta) > 0 \\ & \text{or if} \quad k=1 \text{ and } \beta=0 \\ = S_M(\beta) & \text{if} \quad -(D/v_2)v_1 \leq -DA(\beta)K(\beta) \leq 0 \\ = (D/v_2)v_1 & \text{if} \quad -DA(\beta)K(\beta) \leq -(D/v_2)v_1 \end{cases}$$

and

$$\delta^3 \begin{cases} = (D/v_2)v_1 & \text{if} \quad -(D/v_2)v_1 \leq -DA(\gamma)K(\gamma) \\ & \text{or if} \quad k=1 \text{ and } \gamma=0 \\ = S_M(\gamma) & \text{if} \quad -(D/v_2)v_1 > -DA(\gamma)K(\gamma) \ . \end{cases}$$

We have now proved:

Lemma 6: Let X and Y be two pseudo-airways having a Class I relationship. Let ① be an arbitrary aircraft on pseudo-airway X, and let ② be the nearest upstream aircraft on airway Y to ① as ① crosses the intersection. Then ① and ② will generate a crossing conflict exactly when

$$D < MC,$$

where

D $\equiv$ the distance between ① and ② as ① crosses the intersection

M $\equiv$ the minimum separation permitted between aircraft

and

$$
C = \begin{cases}
1 & \text{if } \delta = \delta^2 \text{ and } A(\beta)K(\beta) < 0 \\
 & \text{or if } \delta = \delta^2 \text{ and } k=1 \text{ and } \beta=0 \\
 & \text{or if } \delta = \delta^3 \text{ and } k=1 \text{ and } \gamma=0 \\
v_2 / v_1 & \text{if } \delta = \delta^2 \text{ and } A(\beta)K(\beta) \geq (v_1 / v_2) \\
 & \text{or if } \delta = \delta^3 \text{ and } (v_1 / v_2) \geq A(\gamma)K(\gamma) \\
[\,(K(\theta)A(\theta))^2 (1 + k^2) + 1 + 2K(\theta)A(\theta)\,(\cos\theta - k - kK(\theta)A(\theta)\cos\theta)\,]^{-1/2} & \text{otherwise, where}
\end{cases}
$$

$$
\theta = \begin{cases} \beta & \text{if } \delta = \delta^2 \\ \gamma & \text{if } \delta = \delta^3 . \end{cases} \qquad (2\text{-}23)
$$

Theorem 6: Assume the same hypothesis as in Lemma 6. Let S_Y be the expected separation between aircraft on pseudo-airway Y. Then ① and ② will generate a crossing conflict with probability

$$
PCON_{X,Y} = \begin{cases} 0 & \textit{if} \quad C < 1 \\ 1 - exp\left(- \dfrac{CM - M}{S_Y - M} \right) & \textit{otherwise,} \end{cases} \qquad (2-24)
$$

where C is as in Lemma 6.

Proof: The key to this proof is the assumption that the distance D between ① and ② as ① crosses the intersection is distributed according to a delayed negative exponential distribution with delay M and mean S_Y. This assumption is appropriate here for the same reason we discussed in section 2.1.1 : a separation less than M would mean that the controller had allowed an overtaking conflict to occur before ① ever reached the intersection. Remember, we are computing the probability that ① generates a crossing conflict with an aircraft on Y.

Given the assumption, the result follows directly. If $C < 1$, then ① and ② generate a crossing conflict only if $D < M$. By the above discussion, $D \geq M$ always holds, so $PCON_{X,Y} = 0$.

If, on the other hand, $C \geq 1$, then $PCON_{X,Y}$ is simply the probability that $M \leq D \leq CM$. By the definition of the delayed negative exponential distribution, this is simply

$$PCON_{X,Y} = 1 - exp\left(- \frac{CM - M}{S_Y - M} \right)$$

■

CLASS II: If Y is in Class II relative to X , then both ① and ② fly the same track outbound from the intersection, as in Figure 2-17.

If ① and ② have not conflicted by the time ① reaches the intersection, then any conflict between them will be defined to be an overtaking conflict. Thus, to calculate the probability that ① and ② will generate a crossing conflict, we need consider only

Case 1: Both ① and ② are inbound

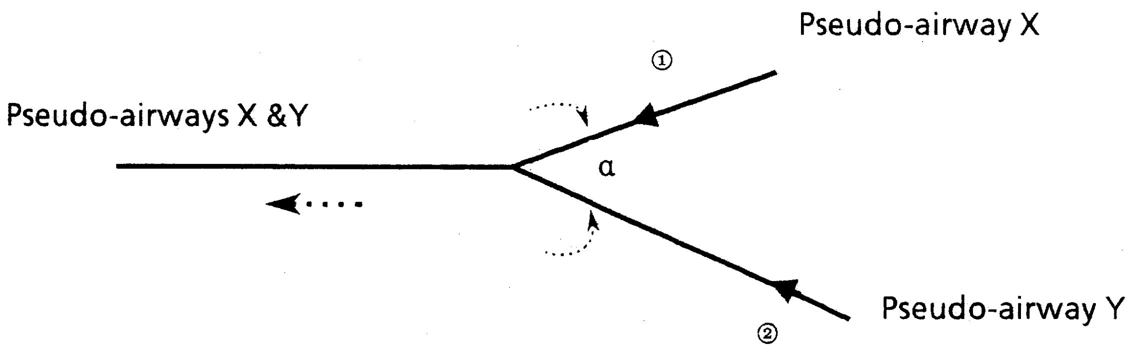

Figure 2-17

Two Pseudo-airways with a Common Outbound Track

We will use the same variables defined in our discussion under Class I.

<u>Case 1</u>: Both ① and ② are inbound.

Let δ^1 be the minimum distance between ① and ② under Case 1 (i.e. while both are inbound). Remember that ② is defined to be the upstream aircraft on Y that is closest to the intersection as ① crosses the intersection. If X and Y were straight airways intersecting at angle α, then ① would be $-DA(\alpha)K(\alpha)$ units away from the intersection when the separation between ① and ② was minimal (we showed this in the proof of Theorem 1). If $-DA(\alpha)K(\alpha) \leq 0$, then the separation (call it d) between ① and ② is monotonically decreasing as long as ① is inbound. This would mean that $\delta^1 = D$, since $d = D$ when ① crosses the intersection. If, on the other hand, $-DA(\alpha)K(\alpha) \geq 0$, then d is minimal sometime prior to the time ① reaches the intersection; d's mminimal value in this situation is just $S_M(\alpha)$ from the proof of Theorem 1. Thus

$$\delta = \delta^1 \begin{cases} = D & \text{if } -DA(\alpha)K(\alpha) \leq 0 \\ & \text{or if } k=1 \text{ and } \alpha=0 \\ = S_M(\alpha) & \text{otherwise.} \end{cases} \tag{2-25}$$

This gives us

Lemma 7: Let X and Y be two pseudo-airways having a Class II relationship. Let ① be an arbitrary aircraft on pseudo-airway X, and let ② be the nearest aircraft to ① that is upstream on Y at the instant ① crosses the intersection. Then ① and ② will generate a crossing conflict exactly when $D < MC$, where

$D \equiv$ the distance between ① and ② as ① crosses the intersection

$M \equiv$ the minimum separation permitted between aircraft

and

$$C \begin{cases} = 1 & \text{if } \quad -A(\alpha)\,K(\alpha) \leq 0 \\ & \text{or if } \quad k=1 \text{ and } \alpha=0 \\ = S_M(\alpha) & \text{otherwise.} \end{cases}$$

Proof: The two possible values for C again correspond to the two possible values for δ.

■

Theorem 7: Assume the hypothesis as in Lemma 7. Let S_Y be the expected separation between aircraft on pseudo-airway Y. Then ① and ② will generate a crossing conflict with probability

$$PCON_{X,Y} = 1 - \left(\frac{S_Y - M}{S_Y}\right) exp\left(\frac{M - CM}{S_Y - M}\right) \tag{2-26}$$

where C is as defined as in Lemma 7.

Proof: The proof of this theorem directly parallels the development of Lemma 2 and Theorem 2. We define a null zone on pseudo-airway Y as any segment of length M behind an aircraft on Y. Null zones are those portions of airway Y which are empty of aircraft due to the delayed negative exponential distribution of inter-aircraft distances. We then show that the probability that ① crosses the intersection in a null zone on Y is M/S_Y, just as we did in Lemma 2. Finally, we compute the probability that ② is within CM of ① as ① crosses the intersection by looking at two cases: when ① does, and does not, hit a null zone on Y as ① crosses the intersection. The result follows directly.

The reader should note the one significant difference between Theorems 2 and 7 : C is defined in Theorem 2 to include Case 3 (where ① and ② are both outbound); in Theorem 7 only Cases 1 and 2 are considered.

■

CLASS III: If Y is in Class III relative to X, then the two pseudo-airways meet only at the intersection. This is simply the case discussed leading up to Theorem 3 , so the following theorem follows directly from Theorem 4:

Theorem 8: Let X and Y be two pseudo-airways which meet only at the intersection. Then the probability that an arbitrary aircraft on X conflicts with an aircraft upstream on Y is just

$$PCON_{X,Y} = 1 - \left(\frac{S_Y - M}{S_Y} \right) exp\left(\frac{M - CM}{S_Y - M} \right) \qquad (2-27)$$

where S_Y is the expected distance between aircraft on Y , and C is defined as in Corollary 3.1.

■

Notice that only crossing conflicts occur when Class III applies.

We are now ready to compute the controller intervention rate due to crossing conflicts at an arbitrary intersection (Figure 2-18).

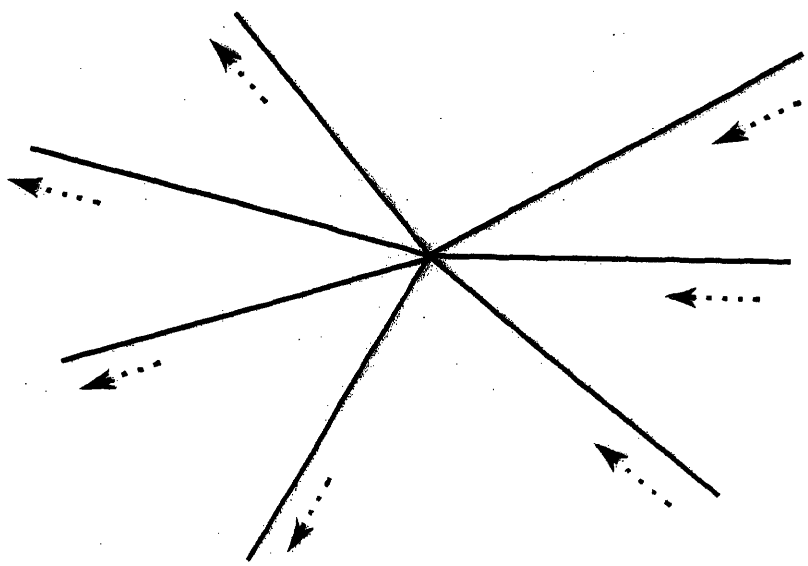

Figure 2-18

An Intersection with Airway Changes

Remember that aircraft are permitted to change airways at the intersection, and that aircraft on a given pseudo-airway all travel at the same constant airspeed.

Assume that the intersection in Figure 2-18 is decomposed into N pseudo-airways. Remember that each pseudo-airway represents a distinct inbound/outbound leg pair. Let λ_i be the traffic rate on pseudo-airway i, and v_i the airspeed on that same pseudo-airway. Then we can use Theorems 6 - 8 to compute $PCON_{i,j}$ for each pseudo-airway pair (i,j). For an arbitrary aircraft ① on pseudo-airway i we can then calculate

$$\prod_{\substack{j=1 \\ j \neq i}}^{N} \left(1 - PCON_{i,j}\right) , \qquad (2-28)$$

the probability that ① will pass through the intersection without a crossing conflict. This, in turn, gives

$$PCON_i = 1 - \prod_{\substack{j=1 \\ j \neq i}}^{N} \left(1 - PCON_{i,j}\right) , \qquad (2-29)$$

the probability that ① <u>will</u> conflict with an aircraft upstream on one of the other $N-1$ pseudo-airways. We can then compute

$$\lambda_i PCON_i \quad , \qquad (2-30)$$

the rate at which aircraft on pseudo-airway i are involved in crossing conflicts with upstream aircraft on other pseudo-airways. Finally, we get

$$R_c = \sum_{i=1}^{N} \lambda_i PCON_i$$

$$= \sum_{i=1}^{N} \lambda_i \left[1 - \prod_{\substack{j=1 \\ j \neq i}}^{N} \left(1 - PCON_{i,j}\right)\right] \qquad (2-31)$$

the intervention rate at the intersection due to crossing conflicts.

We have now proved:

Theorem 9: Consider an intersection as shown in Figure 2-18, which can be decomposed into N pseudo-airways. Let each pseudo-airway i have traffic rate λ_i ; the aircraft travel at constant velocity v_i.

Then R_c, the controller intervention rate due to crossing conflicts, at this intersection is

$$R_c = \sum_{i=1}^{N} \lambda_i PCON_i$$

$$= \sum_{i=1}^{N} \lambda_i \left[1 - \prod_{\substack{j=1 \\ j \neq i}}^{N} \left(1 - PCON_{i,j}\right)\right] \qquad (2-32)$$

where the $PCON_{i,j}$ are as defined in Theorems 6, 7, and 8.

2.1.6 Arbitrary Airspeed Distributions

Now let's consider the case where the aircraft airspeed distribution on each airway is no longer a single point mass. Let the airspeed at the entry point on pseudo-airway i be distributed according to an arbitrary pdf $f_i(v)$. The separation between successive aircraft on i at the airway's entry point is distributed according to a delayed negative exponential distribution with delay M and mean S_i. Each aircraft on pseudo-airway i travels at a constant velocity, whose value at the entry point is distributed according to pdf $f_i(x)$.

Notice that the rate r_i at which aircraft enter pseudo-airway i is

$$r_i = \frac{1}{S_i} \int_0^{\infty} x f_i(x) \; dx \qquad (2-33)$$

where

$$\int_0^{\infty} x f_i(x)\ dx$$

is just the expected value of the velocity distributed according to $f_i(x)$.

Lets first consider only those aircraft on each pseudo-airway i (for $i = 1,2,\ldots,N$) which have velocity $v_i + dv$, where each $v_i > 0$ and dv is very small. Then by Theorem 9 we can calculate the controller intervention rate generated by these aircraft:

$$R_c(v_1, v_2, \ldots, v_N) = \sum_{i=1}^{N} r_i f_i(v_i)\, dv\ PCON_i(v_1, v_2, \ldots, v_N) \tag{2-34}$$

This follows since $r_i f_i(v_i)\,dv$ is the rate at which aircraft enter airway i with an airspeed in the range $[v_i, v_i + dv]$. Paralleling the argument leading up to Theorem 5,

$$PCON_i(v_1, v_2, \ldots, v_N) = 1 - \prod_{\substack{j=1 \\ j \neq i}}^{N} (1 - PCON_{i,j}) \tag{2-35}$$

and each $PCON_{i,j}$ is computed using v_i and v_j, respectively, for the airway velocities, and

$$\frac{v_i}{r_i f_i(v_i)\, dv}$$

for the expected separation on each airway i.

Once we have $R_c(v_1, v_2, \ldots, v_N)$, we "simply" integrate over all values of the v_i to get the total controller intervention rate:

$$R_c = \int_0^{\infty} \cdots \int_0^{\infty} R_c(v_1, v_2, \ldots, v_N)\, dv_1 \ldots dv_N \tag{2-36}$$

Of course, R_c will in general be very difficult to compute exactly. Various methods of approximation may be used; at this point we will mention only that discrete distributions $f_i(x)$ may make the numerical evaluation of R_c fairly straight-forward.

2.2 Controller Interventions Due to Overtaking Conflicts

2.2.1 The Basic Model - A Single Airway

The purpose of this section is to generalize the model to include controller interventions caused by overtaking traffic. We begin by looking at a simple case: a single airway segment A with constant ground track.

Figure 2-19

Overtaking Traffic on a Single Airway Segment

Let the length of the airway be L , and let the airspeeds of the aircraft entering the airway from the right be distributed according to pmf $f_V(x)$, where $f_V(x)$ is a discrete function - that is, a series of point masses. We want to compute the probability that aircraft ①, entering from the right, will traverse the entire length of the airway without conflicting with a following aircraft. Notice that an overtaking conflict occurs when a following aircraft ② pulls within M of ①; that is, when ① and ② have violated the minimum separation standard. Initially, we'll assume that all aircraft enter the airway without having to change heading; that is, their ground track just prior to the entry point into A is the same as the track along A.

Now consider aircraft ① having airspeed v_1, as it enters A. We want to compute the probability that ① will traverse the entire length of A without being overtaken (within M) by an aircraft having airspeed v_2. To compute this probability we will need to know the minimum separation between ① and the nearest following aircraft having speed v_2 (call it aircraft ②), that will guarantee that ① and ② remain separated by at least M until ① reaches the end of A. Lets call this minimum separation $d(v_1, v_2)$.

Now if $v_1 \geq v_2$, then ① and ② will never be closer than they are when ① is at A's entry point. Thus

$$d(v_1, v_2) = M \qquad \text{for } v_1 \geq v_2.$$

If, on the other hand, $v_1 < v_2$, ② will close the distance between them during ①'s flight along A. In fact, he will close at a rate $(v_2 - v_1)$. Since the time ① spends on A is L/v_1, the total decrease in separation will be

$$\frac{(v_2 - v_1)L}{v_1} .$$

Since we want their final separation to be at least M, we want

$$M \leq d(v_1, v_2) - \frac{(v_2 - v_1)L}{v_1} . \qquad (2-37)$$

Thus the minimum separation between ① and ② at ①'s entry which will insure no overtake, is

$$d(v_1, v_2) = M + \frac{(v_2 - v_1)L}{v_1} . \qquad (2-38)$$

Now let S be the expected separation between aircraft as they pass the entry point onto A. Then the expected distance between two aircraft having velocity v_2 is

$$S_{v_2} = \frac{S}{f_V(v_2)} ,$$

and thus the separation between ① and ② as ① enters A is

$$\sim DNE\,(M, 1/[S_{v_2} - M]) .$$

This gives the probability that ② will not overtake ① as

$$P_{NO}(v_1, v_2) = \int_{d(v_1, v_2)}^{\infty} \frac{1}{S_{v_2} - M} \exp\left(- \frac{x - M}{S_{v_2} - M} \right) dx$$

$$= \int_{M + \frac{(v_2 - v_1)L}{v_1}}^{\infty} \frac{1}{S_{v_2} - M} \exp\left(- \frac{x - M}{S_{v_2} - M} \right) dx$$

$$= \exp\left(- \frac{\frac{(v_2 - v_1)L}{v_1}}{S_{v_2} - M} \right) . \qquad (2-39)$$

Therefore

$$P_{NO}(v_1, v_2) \begin{cases} = 1 & \text{if } v_1 \geq v_2 \\ = \exp\left(\frac{(v_1 - v_2)L}{v_1 (S_{v_2} - M)} \right) & \text{if } v_1 < v_2 \end{cases} \qquad (2-40)$$

We use the subscript NO to indicate "no overtake."

Now lets consider the case when ② approaches the entry point to A along a different ground track: that is, ② will need to change ground track at the entry. This situation is depicted graphically in Figure 2-20.

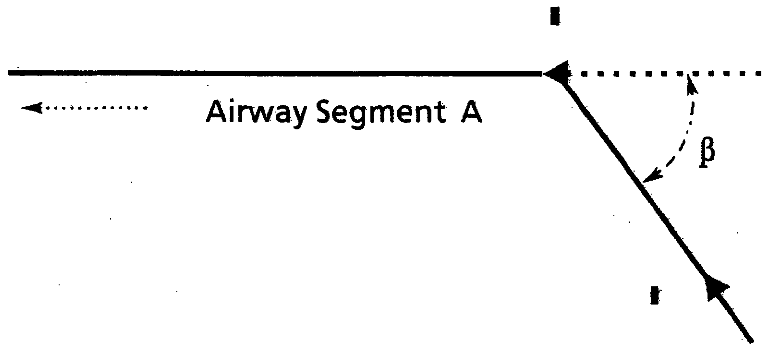

Figure 2-20
Overtaking When Ground Track Changes at the Entry Point

To see why this is of concern, consider a case where $v_1 = v_2$ and the separation between ① and ② is exactly M. Assume β (the track change made by ②) is not zero. Then when ① is at the entry point, ① and ② do not conflict. Similarly, after ② has passed the entry point they will not conflict. In both cases their separation will be exactly M. <u>However</u>, starting at the time ① enters A, and until ② enters, a conflict will exist: their separation will be less than M, due to their non-zero relative velocities. If ① did not approach A along ②'s ground track, then this conflict was considered (and counted) as a crossing conflict in section 1.A. But if ① and ② did approach A on the same track, then their momentary conflict has not yet been counted by the model as it stands. To do so, we will include this case as a type of overtake on airway segment A.

Lets first consider the case where $v_1 \geq v_2$. We saw in section 2.1 that if ① and ② are separated by a distance D as ① enters A, the minimum separation between ① and ② during the time between ① and ②'s entries onto A will be

$$\delta \begin{cases} = D & \text{if} \quad -DA(\beta)K(\beta) > 0 \\ = S_M(\beta) & \text{if} \quad -(D/v_2)v_1 \leq -DA(\beta)K(\beta) \leq 0 \end{cases} \qquad (2\text{-}41)$$

$$= \quad D(v_1/v_2) \quad \text{if} \quad -DA(\beta)K(\beta) < -(D/v_2)v_1 .$$

Since $v_1 \geq v_2$, the third case ($-DA(\beta)K(\beta) < -(D/v_2)v_1$) will not occur. Thus

$$\delta \begin{cases} = D & \text{if} \quad -DA(\beta)K(\beta) > 0 \\ = S_M(\beta) & \text{otherwise.} \end{cases} \tag{2-42}$$

We want $\delta \geq M$, so we want

$$D \geq M \quad \text{if} \quad DA(\beta)K(\beta) < 0 \text{, and}$$

$$D \geq M\left(\frac{S_M(\beta)}{D}\right)^{-1} \quad \text{otherwise,}$$

where, as before

$$S_M(\beta) \equiv \left((-DA(\beta)K(\beta))^2(1 + k^2) + D^2 + 2D^2A(\beta)K(\beta)(cos\beta - k - kK(\beta)A(\beta)cos\beta)\right)^{\frac{1}{2}}$$

and

$$\frac{S_M(\beta)}{D} \equiv \left((-A(\beta)K(\beta))^2(1 + k^2) + 1 + 2A(\beta)K(\beta)(cos\beta - k - kK(\beta)A(\beta)cos\beta)\right)^{\frac{1}{2}} .$$

Thus, combining the results of the above discussions, we get for $v_1 \geq v_2$

$$d(v_1, v_2) \begin{cases} = M & if \quad A(\beta)K(\beta) < 0 \\ = M\left(\dfrac{S_M(\beta)}{D}\right)^{-1} & otherwise. \end{cases} \qquad (2-43)$$

Now lets consider the possibility that $v_1 < v_2$. In this case it is conceivable that $v_2 - v_1$ will be so small, and A so short, that ② may pull within M of ① during the time between their respective entries onto A, and yet not be able to close again after ② enters A. This is an unlikely situation in real life, but we include it here in the interests of completeness. Thus for $v_1 < v_2$

$$d(v_1, v_2) = max\left\{ M + \frac{(v_2 - v_1)L}{v_1},\ M\left(\frac{S_M(\beta)}{D}\right)^{-1} \right\}; \qquad (2-44)$$

the first value in brackets represents ② overtaking ① after ② reaches the entry point, while the second is the (unlikely) situation that overtake occurs only between ① and ②'s entry times onto A.

We can now compute $P_{NO}(v_1, v_2)$ to include the possibility that ① and ② conflict between their respective entry times onto A. We get

For $v_1 \geq v_2$:

$$P_{NO}(v_1, v_2) = \begin{cases} 1 & \text{if } \beta = 0 \\ & \text{or if } \beta \neq 0 \text{ and } A(\beta)K(\beta) < 0 \\ \exp\left(\dfrac{M\left(1 - \left(\dfrac{S_M(\beta)}{D}\right)^{-1}\right)}{S_{v_2} - M} \right) & \text{if } \beta \neq 0 \text{ and } A(\beta)K(\beta) \geq 0. \end{cases} \tag{2-44}$$

The second value only applies when, in addition, ① and ② approach A along the same ground track.

For $v_1 < v_2$:

$$P_{NO}(v_1, v_2) = exp\left(\frac{M - d(v_1, v_2)}{S_{v_2} - M} \right) \qquad (2-45)$$

where

$$d(v_1, v_2) = \begin{cases} M + \dfrac{(v_2 - v_1)L}{v_1} & \textit{if ① and ② approach A along different ground tracks} \\ max\left\{ M + \dfrac{(v_2 - v_1)L}{v_1} , \mathrm{M}\left(\dfrac{\mathrm{S_M}(\beta)}{\mathrm{D}} \right)^{-1} \right\} & otherwise. \end{cases} \qquad (2-46)$$

To summarize the results of this section, we state

Definition: Given an airway segment A as in Figure 2-19. An aircraft ② is said to overtake ① on A if ② approaches ① to within M while ① and ② are both on A. If, in addition, ① and ② approach the entry to A along the same tracks, then ② is said to overtake ① on A if their separation is less than M anytime ① is on A.

Theorem 10: Given airway segment A of length L. Let aircraft ① have velocity v_1, and let aircraft ② be the next aircraft of velocity v_2 to arrive at the entry to A after ①. Let S_{v_2} be the expected separation between aircraft of velocity v_2 on A. If ① and ② approach A along different ground tracks, then the probability $P_{NO}(v_1, v_2)$ that ① will not be overtaken by ② on A is

$$P_{NO}(v_1, v_2) \begin{cases} = 1 & \text{if } v_1 \geq v_2 \\ = exp\left(\dfrac{(v_1 - v_2)L}{v_1(S_{v_2} - M)}\right) & \text{otherwise.} \end{cases} \tag{2-47}$$

If ① and ② both approach A along a track at angle β to the ground track of A (see Figure 2-20), then for $v_1 \geq v_2$

$$= \quad 1 \qquad if \quad \beta = 0$$

$$or\, if \qquad \beta \neq 0 \quad and \quad A(\beta)K(\beta) < 0$$

$$P_{NO}(v_1, v_2) \tag{2-48}$$

$$= exp\left(\frac{M\left(1 - \left(\frac{S_M(\beta)}{D}\right)^{-1}\right)}{S_{v_2} - M} \right)$$

$$if \quad \beta \neq 0 \quad and \quad A(\beta)K(\beta) \geq 0$$

while for $v_1 < v_2$

$$P_{NO}(v_1, v_2) \quad = \quad exp\left(\frac{M - d(v_1, v_2)}{S_{v_2} - M} \right) \tag{2-49}$$

where

$$d(v_1, v_2) \quad = \quad max\left\{ M + \frac{(v_2 - v_1)L}{v_1} \; , \; M\left(\frac{S_M(\beta)}{D} \right)^{-1} \right\} \tag{2-50}$$

■

Now that we have $P_{NO}(v_1, v_2)$ we can compute the rate at which controller interventions will occur due to overtakes on airway segment A. Let

$$E \equiv \{x \mid f_V(x) \neq 0\};$$

that is, E is the set of possible velocities on A.

Then the probability $P_{NO}(v_1)$ that an aircraft of velocity v_1 will not be overtaken on A is

$$P_{NO}(v_1) = \prod_{v \in E} P_{NO}(v_1, v).$$

The rate $R_o(v_1)$ at which aircraft of velocity v_1 will generate overtaking interventions on A will be

$$R_o(v_1) = \lambda f_V(v_1)\Big(1 - P_{NO}(v_1)\Big), \tag{2-51}$$

where λ is the expected traffic density (aircraft/unit time) at the entry point to A.

Finally, the total rate of controller interventions due to overtaking conflicts on A will be

$$R_o = \lambda \sum_{v_1 \in E} \Big(f_V(v_1)\Big)\Big(1 - P_{NO}(v_1)\Big). \tag{2-52}$$

2.2.2 Adding Overtaking Situations to the General Crossing Conflict Model

In section 2.1 we computed R_c, the expected controller intervention rate due to crossing conflicts at a generalized enroute intersection. In this section we will outline how one can find R_0, the rate of interventions at the intersection due to overtaking conflicts. We will then argue that the overall controller intervention rate, R, is simply

$$R = R_c + R_0 .$$

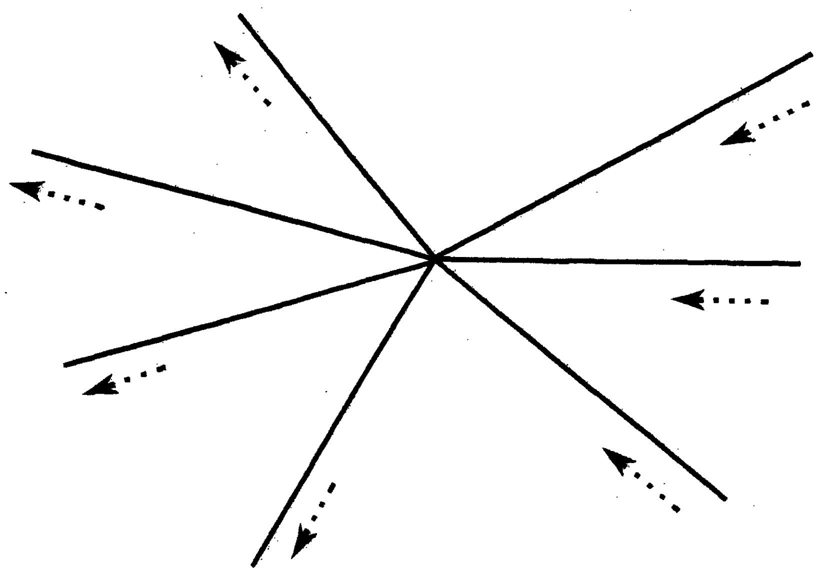

Figure 2-21

A Generalized Airway Intersection

To compute R_0 we note that the intervention rate due to overtaking conflicts for the intersection is simply the sum of the rates for each of the individual simple segments. A simple segment is a portion of the airway consisting of a single center-line with an intersection or a sector boundary at each end. Simple segments do not have intersections or sector boundaries except at their endpoints. Thus the intersection in Figure 2-21 is broken down into simple segments as shown in Figure 2-22.

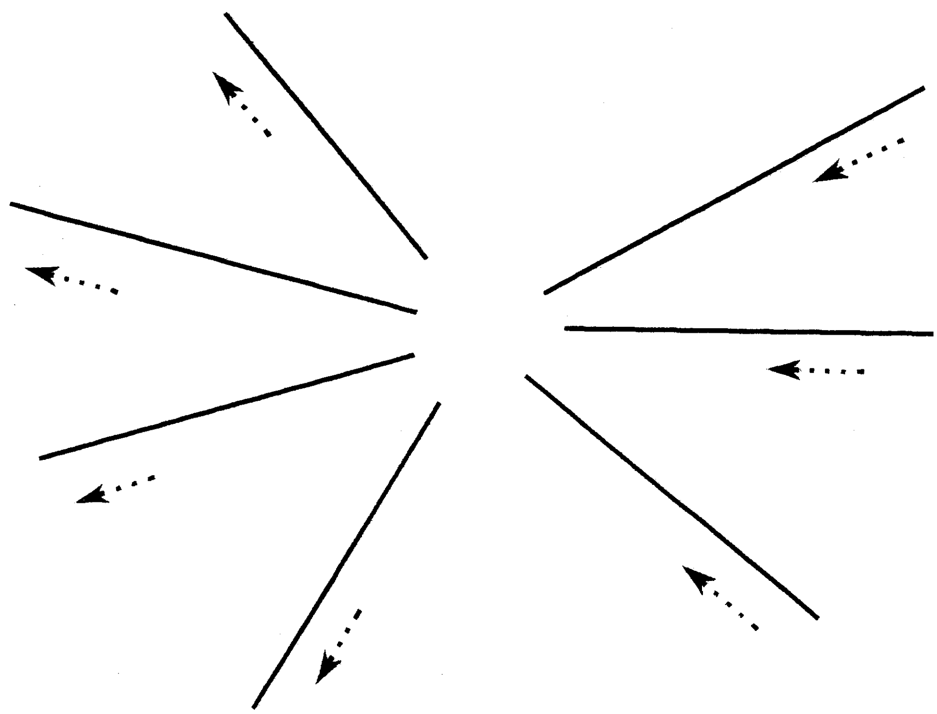

Figure 2-22

An Intersection Broken Into Simple Segments

For each of the simple segments we can compute an intervention rate due to overtaking conflicts. The sum of these intervention rates gives the value for R_0 at the intersection. Note that the intersection must be defined so as to include the lengths of each of the simple segments, since the intervention rate is a function of those values (ie. a longer segment will, ceteris parabus, give a higher rate).

In Section 2.4 we will show how R_0 can be computed for a typical en route airway network.

2.3 Off-Airway Traffic: The Gas Model

Even at higher altitudes, we cannot assume that all traffic will be confined to the published airways. For a variety of reasons, aircraft often request and are granted permission to file "point-to-point", that is, along a path not coinciding with a published route. If we wish to model the total number of controller interventions within a given sector of airspace, we must take the off-airway traffic into consideration.

One way to do this is to break the off-airway traffic into two categories: aircraft which fly along ground tracks used frequently by other off-airway traffic, and those whose paths appear to be essentailly random. We can then model the first group just as we have done with traffic along published airways, by considering their ground track to be a new "airway." The random traffic, on the other hand, can be treated by using the gas model described in the Introduction.

Consider, for example, the situation shown in Figure 2-23. This figure represents a

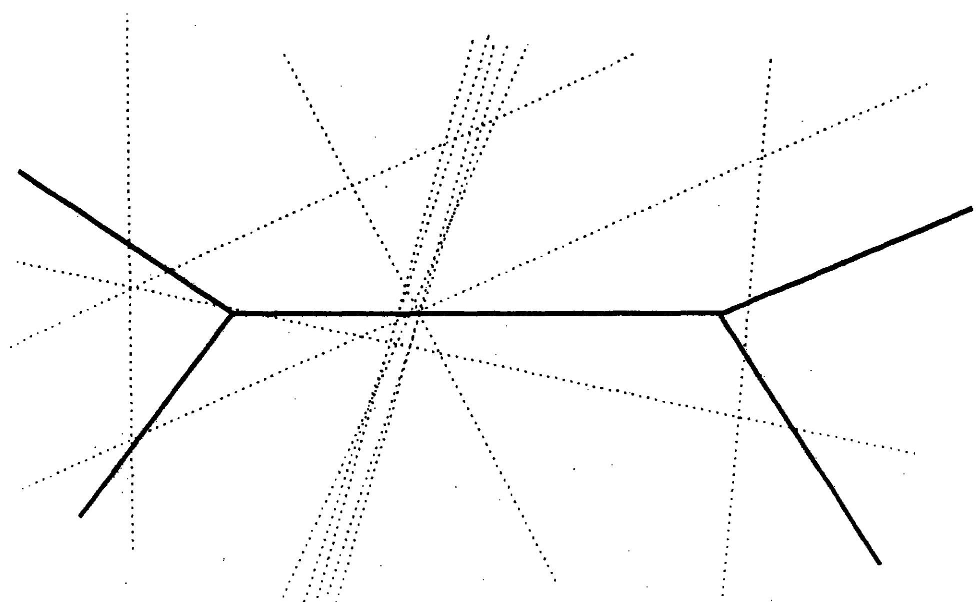

Figure 2-23

Off-Airway Traffic

hypothetical record of all off-airway traffic passing through a specific sector of airspace during a one hour period. The solid lines represent published airways, while each dotted line represents a single off-airway flight through the sector. Notice that while many of the tracks seem to be randomly distributed throughout the sector, several are clustered around a single path running northeast - southwest through the center of the sector.

While one could attempt to model <u>all</u> the off-airway traffic as if it were randomly distributed, it might be more accurate (particularly if the clustering effect seen in Figure 2-23 is repeated in other observations) to model the clustered traffic as if it were following an actual published airway. In other words, we would model the sector as if it were structured as in Figure 2-24. The

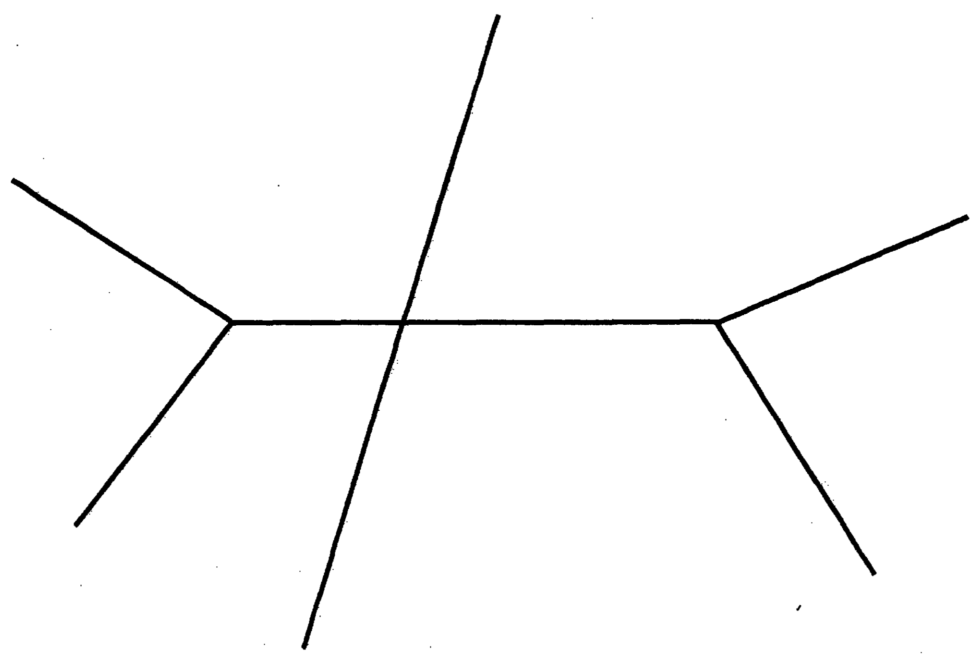

Figure 2-24

An Augmented Network

traffic densities and airspeed distribution for the new airway would be determined by the nature of the clustered traffic from Figure 2-23.

Now we need only superimpose on the augmented network the random traffic from Figure 2-23. To do this we can adapt the gas model to determine the additional intervention rate generated by the random traffic.

Lets assume that the airway network has an expected traffic density of N aircraft/mile^2 ; that is, N is the expected number of aircraft on airways in the sector divided by the area of the sector. Now let ① be a random point-to-point aircraft traversing the sector at airspeed v.

In Endoh [1982] and Endoh and Odoni [1983] we see that the expected relative velocity V_{rel} between two off-airway aircraft traveling at velocity v is

$$V_{rel} = \frac{4v}{\pi} , \qquad (2-53)$$

assuming the ground tracks of each are randomly distributed throughout the sector. Since the airspeeds found at a given altitude in the en route structure tend to cluster tightly around a central value, we will use

$$\frac{4\sqrt{v_1 v_2}}{\pi} \qquad (2-54)$$

to approximate the relative velocity between random aircraft of airspeeds v_1 and v_2, when v_1 and v_2 differ by no more than 10%. For differences greater than 10%, we can use a simple program to numerically integrate

$$V_{rel}(v_1, v_2) = \frac{1}{2\pi}\int_0^{2\pi} (v_1^2 + v_2^2 - 2v_1 v_2 \cos \alpha)^{\frac{1}{2}} \, d\alpha , \qquad (2-55)$$

which is the exact expression for the expected relative velocity between two aircraft of speed v_1 and v_2 whose ground tracks are randomly distributed across the interval $[0, 2\pi]$.

To show why

$$\frac{4\sqrt{v_1 v_2}}{\pi}$$

is a good approximation for $V_{rel}(v_1, v_2)$ when $v_1 - v_2$ is small, let

$$g \equiv \sqrt{v_1 v_2} .$$

Then

$$(v_1^2 + v_2^2 - 2v_1 v_2 \cos\alpha)^{\frac{1}{2}} = \left((g+\delta_1)^2 + (g+\delta_2)^2 - 2g^2 \cos\alpha\right)^{\frac{1}{2}} \qquad (2-56)$$

where, of course, $g + \delta_1 = v_1$ and $g + \delta_2 = v_2$.

So

$$(v_1^2 + v_2^2 - 2v_1 v_2 \cos\alpha)^{\frac{1}{2}}$$

$$= (g^2 + 2g\,\delta_1 + \delta_1^2 + g^2 + 2g\,\delta_2 - 2g^2 \cos\alpha)^{\frac{1}{2}}$$

$$= (2g^2 - 2g^2 \cos\alpha + 2g(\delta_1 + \delta_2) + \delta_1^2 + \delta_2^2)^{\frac{1}{2}}$$

$$\approx \left(2g^2(1-\cos\alpha)\right)^{\frac{1}{2}} \qquad \text{for small } \delta_1 \text{ and } \delta_2$$

$$= g\sqrt{2(1-\cos\alpha)} \qquad . \qquad (2-57)$$

Therefore

$$V_{rel}(v_1, v_2) = \frac{1}{2\pi}\int_0^{2\pi} (v_1^2 + v_2^2 - 2v_1 v_2 \cos\alpha)^{\frac{1}{2}}\, dx$$

$$\approx \frac{1}{2\pi}\int_0^{2\pi} g\sqrt{2(1-\cos\alpha)}\, dx$$

$$= \frac{4\sqrt{v_1 v_2}}{\pi} \qquad . \qquad (2-58)$$

In Table 2-7 we see some comparisons of $V_{rel}(v_1, v_2)$ versus $[4(v_1 v_2)^{1/2}]/\pi$ for velocities typical of the high altitude en route structure. Notice that when v_1 and v_2 differ by less than 10%, the approximation $[4(v_1 v_2)^{1/2}]/\pi$ is well within 1% of $v_{rel}(v_1, v_2)$.

Velocity		$V_{rel}(v_1,v_2)$	Approx-imation	% Difference
v_1	v_2			
350	360	452.2	452.0	.06
440	460	573.6	572.8	.14
340	360	446.4	445.5	.22
440	480	587.9	585.1	.47
325	360	438.3	435.5	.63
400	500	583.9	569.4	2.48
300	360	425.9	418.4	1.75

Table 2-7

Following Endoh [1982], we find the collision rate between aircraft of velocities v_1 and v_2 is

$$C(v_1, v_2) = MV_{rel}(v_1, v_2)\, \rho(v_1)\, \rho(v_2) \qquad (2-59)$$

where $V_{rel}(v_1,v_2)$ is the expected relative velocity between aircraft of speeds v_1 and v_2, and $\rho(v_i)$ is the density (aircraft/unit area) for aircraft of velocity v_i.

In general we will need to figure a conflict rate for two cases:

(1) When aircraft of speed v_1 are off-airway and aircraft of speed v_2 are (modeled) on airway.

(2) When both airspeeds are for off-airway traffic.

The conflict rates derived from case (2) will need to be multiplied by a factor of 0.5, to avoid double counting conflicts between off-airway aircraft.

<u>Theorem 11</u>: **Consider an en route sector as in Figure 2-24, consisting of both on and off-airway traffic. The conflict rate between on and off-airway aircraft is**

$$C^A = \sum_{v_1 \in F} \sum_{v_2 \in E} MV_{rel}(v_1,v_2)(N^R_{v_1})(N^A_{v_2}) \qquad (2-60)$$

where

F = the set of airspeeds flown by off-airway traffic

E = the set of airspeeds flown by on-airway traffic

M = the minimum acceptable separation

$V_{rel}(v_1,v_2)$ = the expected relative velocity between an off-airway aircraft of speed v_1 and an on or off-airway aircraft of speed v_2

$N_{v_1}{}^R$ = the average density (number of aircraft/unit area) of off-airway traffic with speed v_1

$N_{v_2}{}^A$ = the average density (number of aircraft/unit area) over the whole sector, of airway traffic of speed v_2

Similarly, the conflict rate involving two off-airway aircraft is

$$C^R = \frac{1}{2} \sum_{v_1 \in F} \sum_{v_2 \in F} MV_{rel}(v_1,v_2)(N^R_{v_1})(N^R_{v_2}) \quad . \qquad (2-61)$$

The total additional controller intervention rate due to the addition of off-airway traffic to the model is simply

$$R_R = C^A + C^R$$

Proof: The expressions for C^A and C^R are straight forward summations over possible airspeed pairs.

Notice that we are estimating controller interventions by counting conflicts , contrary to our earlier practice. The density of random , off-airway traffic will be such that this approximation should have little effect on the results.

■

2.4 An Example

In this section we provide an example of how the model developed in Sections 2.1 - 2.3 can be applied to a typical high altitude airway sector. The network we will use is shown in Figure 2-25; it is representative of a single controller's area of responsibility within the airspace assigned to an Air Route Traffic Control Center.

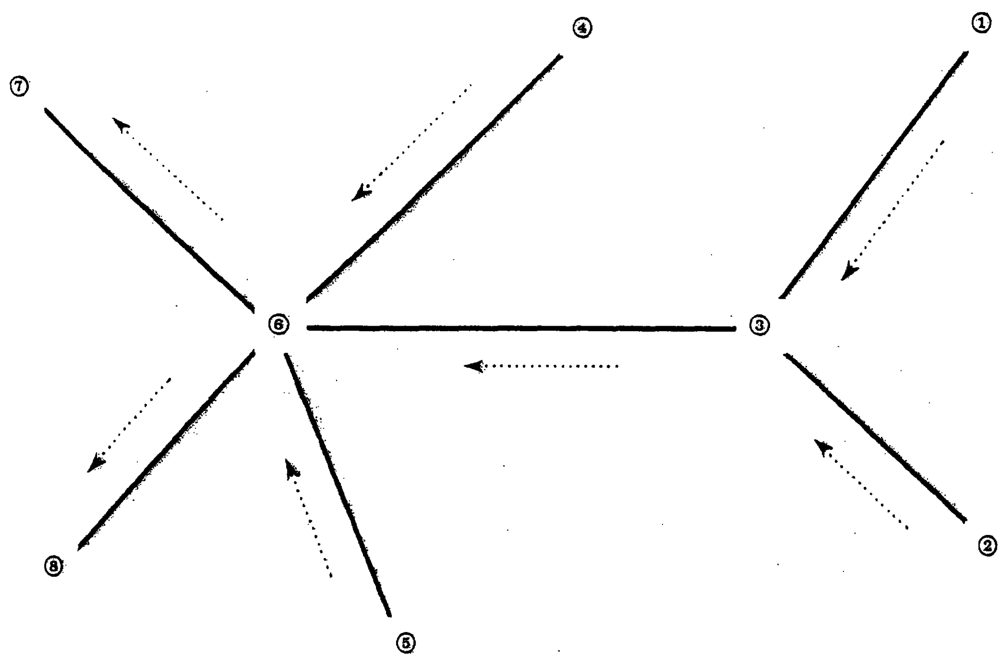

Figure 2-25

A Typical High Altitude Sector

We will assume that this is a two-dimensional sector, so all traffic will be treated as if it were at exactly the same altitude. The direction of flow of traffic along the network is shown in Figure 2-25, as are the labels assigned to each of the eight nodes. Each of the airway segments and pseudo-airways will be named according to the nodes they encompass. Thus the segment from node ③ to node ⑥ will be segment (or arc) 3-6; the pseudo-airway from node ④ through node ⑥ to node ⑦ will be called pseudo-airway 4-6-7. The structure of the network (ie. length and ground track of each segment) will remain fixed throughout this example: these values are shown in Table 2-8.

Airway Segment	Length (miles)	Ground Track
1-3	50	225°
2-3	60	320°
3-6	50	270°
4-6	65	235°
5-6	40	330°
6-7	80	315°
6-8	50	210°

Table 2-8
Network Parameters

Program 2 in Appendix 3 applies the model developed in preceding sections to the network in Figure 2-25. The program is written in BASIC language and runs on a Zenith 150 micro-computer in 1-10 minutes, depending on the complexity of the traffic flow (in particular, the number of different airspeeds per airway segment). Thus the model can be applied to a real-world analysis of intervention rates without requiring sophisticated (or expensive) software or hardware.

In outline, the program works as follows:

Lines 10-890

Asks the operator to input

1. The number of airspeeds flown on each input arc (arcs 1-3, 2-3, 4-6, 5-6).

2. The value of each distinct airspeed on each input arc.

3. The fraction of traffic on each input arc flying each airspeed.

4. The total traffic density on each input arc.

5. The fraction of traffic on each of arcs 3-6, 4-6, and 5-6 that exits on arc 6-7.

Computes the above parameters for arc 3-6.

Initializes variables.

Lines 900-1220

Computes intervention rate due to crossing conflicts at intersection 3 ($\equiv$RC3)

Uses Class II subroutine at line 1500

Prints RC3

Lines 1270-1490 Class I Subroutine

Computes probability of conflict (PCON) at the intersection of two Class I pseudo-airways, given

intersection structure (α,β,γ) and airspeeds (v1 and v2).

Lines 1500-1650 Class II Subroutine

Computes PCON for intersection of Class II pseudo-airways.

Lines 1690-1920 Class III Subroutine

Computes PCON for intersection of Class III pseudo-airways.

Lines 1940-5300

Computes intervention rate (≡RC6) due to crossing conflicts at intersection 6, and the total intervention rate due to crossing conflicts for the network (≡RC).

Uses Class I, II, and III subroutines.

Computations are made for each pair of pseudo-airways to more clearly expose the program's structure. A more efficient coding would compress these lines into two nested FOR – NEXT loops.

Prints RC6 and RC.

Lines 5330-5450 Overtake Subroutine I

Computes intervention rate due to overtakes along a given segment, when overtaking aircraftdoes not plan to follow overtaken aircraft after exiting segment; computes for v2 overtaking v1.

Inputs include velocities (v1 and v2) , segment length, minimum separation standard (M), anddensity of traffic flying at v1 and v2.

Output is probability of no overtake (PNO).

LInes 5470-5680 Overtake Subroutine II

Computes intervention rate due to overtakes along a given segment, when overtaking and overtaken aircraft follow same track upon exiting segment; computes for v2 overtaking v1.

Same inputs as subroutine I.

Output is probability of no overtake (PNO).

Lines 5700-8770

Computes intervention rate due to overtakes on segments 1-3, 2-3, 3-6, 4-6, and 5-6.

Prints total overtake intervention rate (ROV) and total network intervention rate (RC3 + RC6 + ROV).

As an example of the use of Program 2 on a real-world situation, consider the following data as applied to the two-dimensional en route sector in Figure 2-25:

Traffic entering along airway 1-3

density: 10 aircraft per hour

distribution of airspeeds at the entry point:

p(400) = .25
p(430) = .25
p(460) = .25
p(480) = .25

note: p(v) is the probability that an aircraft entering airway 1-3 will have velocity v.

Traffic entering along airway 2-3

density: 12 aircraft per hour

distribution of airspeeds at the entry point:

$$p(410) = .33$$
$$p(450) = .34$$
$$p(500) = .33$$

Traffic entering along airway 4-6

density: 6 aircraft per hour

distribution of airspeeds at the entry point:

$$p(400) = .33$$
$$p(420) = .16$$
$$p(450) = .17$$
$$p(480) = .34$$

Traffic entering along airway 5-6

density: 5 aircraft per hour

distribution of airspeeds at the entry point:

$$p(380) = .25$$
$$p(420) = .25$$
$$p(450) = .50$$

Departure Route Distribution

p (an aircraft departs intersection 6 along airway 6-7, given that is approached 6 along airway 3-6) = .3

p (an aircraft departs intersection 6 along airway 6-7, given that is approached 6 along airway 4-6) = .5

p (an aircraft departs intersection 6 along airway 6-7, given that is approached 6 along airway 5-6) = .4

Program 2 ran for 5.5 minutes on a Zenith 150 and produced the following results:

Intervention rate due to

1. crossing conflicts at intersection 3	=	2.97 per hour
2. crossing conflicts at intersection 6	=	8.28 per hour
3. overtaking conflicts (entire sector)	=	2.96 per hour
		14.21 per hour

Notice that in this particular example, crossing conflicts outnumber overtaking conflicts by nearly four to one. Of course, this result is highly dependent upon the sector geometry and airspeed distributions: one would expect a wider airspeed variation, for example, to produce more overtakes. For this particular sector, one might suggest structural modifications which tend to reduce crossing conflicts might tend to reduce the overall controller intervention rate. The model developed in this cahpter should be very useful in this sort of investigation, since it can be readily coded using any high level language on relatively inexpensive hardware and still run fast enough to allow an empirical analysis of numerous hypothetical network structures and traffic flow rates.

As a simple example of this sort of analysis, consider the sector in Figure 2-25 with incoming traffic restricted to airways 4-6 and 5-6. This in effect reduces to the simple intersection shown in Figure 2-26.

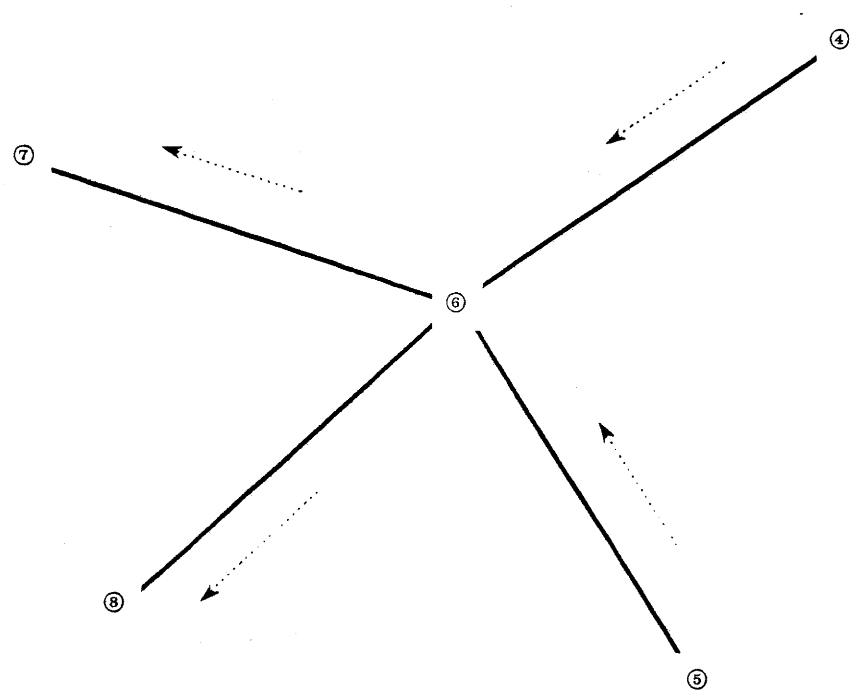

Figure 2-26

Lets assume that traffic entering the sector in Figure 2-26 (on either airway) has a 50% probability of having 400 kt velocity, and a 50% probability of having a 500 kt velocity. Further, let traffic enter both 4-6 and 5-6 at a rate of 10 per hour (that is, ten per hour on each airway). If traffic passing through intersection 6 is equally likely to depart along either 6-7 or 6-8, then Program 2 will compute (25 second run time):

intervention rate due to crossing conflicts	=	3.46 per hour
intervention rate due to overtakes	=	1.29 per hour
total controller intervention rate	=	4.75 per hour

Now lets assume that its possible to restrict all the 500 kt traffic to pseudo-airway 4-6-7, and all the 400 KT aircraft to pseudo-airway 5-6-8. Both airways continue to support a traffic density of 10 aircraft per hour. This is certainly unlikely in a real-world situation, but this may provide some insight into the potential gain derived from segregating by airspeed as much as possible. Program 2 computes:

intervention rate due to crossing conflicts	=	2.50 per hour
intervention rate due to overtakes	=	0 per hour
total controller intervention rate	=	2.50 per hour

As expected, the overtake rate goes to zero. Somewhat more surprising is the fact that interventions due to crossing conflicts <u>also</u> decrease, by over 25%, thus cutting the total intervention rate nearly in half. This unexpected result may be due to the particular sector geometry and airspeeds chosen, but the key point remains: the model is simple enough to allow a real time empirical investigation of the relationship between controller intervention rates, and sector geometry and airspeed distributions, in two-dimensional networks fully as complex as those found in operational en route sectors.

Chapter 3

Validation of the Two-Dimensional Model

3.1 Comparison of the Analytic Model to Results Produced by Monte Carlo Simulation

In Chapter 2 we developed an analytic model which predicted expected controller intervention rates as a function of sector geometry, traffic density, and the distribution of velocities in the sector. In this chapter we will compare these predictions to the results of Monte Carlo simulations designed to replicate the same geometry, traffic density and velocities. The Monte Carlo simulations will provide further evidence for the validity of the analytic model, and in addition will give some indication of the variance to be expected in real-world controller intervention rates about the point estimate generated by the analytic model.

3.1.1 Interventions Due to Crossing Conflicts

Program 4 is a Monte Carlo simulation of the basic airway intersection model developed in Chapter 2. The program is again written in Basic for the Zenith 150 microcomputer. It takes approximately 1-2 minutes to simulate crossing activity at a single intersection during an 8-hour period. The program models an intersection of two airways which do not change ground track. Aircraft on a given airway fly at the same constant airspeed and do not change airways at the intersection.

The complete coding of Program 4 is given in Appendix 3. A brief outline of the structure of the simulation follows:

Lines 10-480 Initialize Variables and Input Data

User inputs parameter values

- angle between airways
- airspeed on each airway
- minimum separation distance
- expected separation on each airway
- simulation run time

The pseudo-random number generator is automatically seeded by the last two digits of the computer clock

The first aircraft arrives at one of the entry points

The critical miss distances (CRITDIST1 and CRITDIST2) are computed according to the results of Theorem 1.

Lines 500-780 Aircraft Arriving at the Entry Point

Updates data base each time an aircraft arrives at the entry point on either airway

Computes the separation between the arriving aircraft and its successor according to a delayed negative exponential distribution

Lines 800-920 Pick Next Critical Time and Aircraft

Determine the time of the next critical event (i.e. an aircraft arrives at an entry point, or an aircraft arrives at the intersection), and the identity of the critical aircraft

Lines 940-1280 Aircraft Arriving at Intersection

Update data base (aircraft leaves the model data base at the intersection, although the critical miss distance is computed to include conflicts which would occur after crossing the intersection

Check to see if controller intervention is required (i.e. is next aircraft on the other airway within the critical miss distance)

Lines 1300-1480

Print observed controller intervention rate (note that interventions aren't counted during the first hour to avoid start-up errors)

The next six tables present the results of ten 8-hour simulation runs and compare them to the controller intervention rates predicted by the analytic model developed in Chapter 2.

--- TEST CASE 1 ---

Angle between airways = 90°

Velocity (Airway #1) = 300 Kts

Velocity (Airway #2) = 540 Kts

Mean distance between aircraft (Airway #1) = 60 NM

Mean distance between aircraft (Airway #2) = 60 NM

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 1.6947 interventions per hour

Observed intervention rates during ten simulated 8-hour periods = 1.6163 interventions per hour

1.2431
2.4922
1.9999
1.6653
2.3726
1.1202
1.4859
1.2485
1.7290

Mean observed rate = 1.6976 interventions per hour

Range about the mean = [− 34% , + 47%]

--- TEST CASE 2 ---

Angle between airways = 60°

Velocity (Airway #1) = 300 Kts

Velocity (Airway #2) = 540 Kts

Mean distance between aircraft (Airway #1) = 60 NM

Mean distance between aircraft (Airway #2) = 60 NM

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 1.4911 interventions per hour

Observed intervention rates during ten simulated 8-hour periods = 1.6176 interventions per hour

1.8705
2.1246
1.4976
1.4973
1.6185
1.3713
0.9989
2.3651
1.2499

Mean observed rate = 1.6211 interventions per hour

Range about the mean = [− 38% , + 46%]

--- TEST CASE 3 ---

Angle between airways = 150°

Velocity (Airway #1) = 360 Kts

Velocity (Airway #2) = 360 Kts

Mean distance between aircraft (Airway #1) = 60 NM

Mean distance between aircraft (Airway #2) = 60 NM

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 3.5212 interventions per hour

Observed intervention rates during ten simulated 8-hour periods = 2.9909 interventions per hour

3.3362

3.7170

4.3520

3.7159

3.8686

3.9921

3.6239

4.3734

3.4012

Mean observed rate = 3.7371 interventions per hour

Range about the mean = [– 20% , + 17%]

--- TEST CASE 4 ---

Angle between airways = 30°

Velocity (Airway #1) = 300 Kts

Velocity (Airway #2) = 540 Kts

Mean distance between aircraft (Airway #1) = 60 NM

Mean distance between aircraft (Airway #2) = 40 NM

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 2.5937 interventions per hour

Observed intervention rates during ten simulated 8-hour periods =	2.8722	interventions per hour
	2.9929	
	2.9971	
	1.6232	
	2.4892	
	2.2486	
	1.7452	
	2.3744	
	1.3636	
	2.2474	
Mean observed rate =	2.2954	interventions per hour

Range about the mean = [− 41% , + 30%]

Observed rate during one 200-hour simulation = 2.7295

--- TEST CASE 5 ---

Angle between airways = 60°

Velocity (Airway #1) = 360 Kts

Velocity (Airway #2) = 360 Kts

Mean distance between aircraft (Airway #1) = 60 NM

Mean distance between aircraft (Airway #2) = 60 NM

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 1.1536 interventions per hour

Observed intervention rates during ten simulated 8-hour periods = 0.6139 interventions per hour

0.8727

0.4459

1.8504

1.1164

0.8738

1.6231

0.9959

0.8722

1.6235

Mean observed rate = 1.0937 interventions per hour

Range about the mean = [– 54% , + 69%]

--- TEST CASE 6 ---

Angle between airways = 30°

Velocity (Airway #1) = 360 Kts

Velocity (Airway #2) = 360 Kts

Mean distance between aircraft (Airway #1) = 60 NM

Mean distance between aircraft (Airway #2) = 60 NM

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 1.0352 interventions per hour

Observed intervention rates during ten simulated 8-hour periods = 0.8693 interventions per hour

1.2490

0.6191

0.6238

1.3670

0.8733

0.9944

1.1209

1.2455

0.8736

Mean observed rate = 0.9836 interventions per hour

Range about the mean = [– 37% , + 39%]

The results of these simulations support the validity of the analytic model. In four of the six test cases the mean observed rate fell within 5% of the predicted rate. In test case two the difference was about 8%, while in case four it was nearly 12% (below the predicted value). A further look at case four, in the form of a single 200-hour simulation, produced an intervention rate 5% above the predicted value. Of the seven mean observed rates (six test cases plus the single 200- hour run on case four), four were higher than predicted and three were lower.

It is important to notice the variation in the observed rates across the ten 8-hour runs in each test case. The range in 8-hour observed rates was typically $\pm$ 30-40% about the mean, and an average of 7 out of 10 deviated by more than 10% from the mean. This simply reinforces a common controller's observation: their workload is rarely constant over any extended time. Periods of intense activity often alternate with periods of relative quiet.

3.1.2 Interventions Due to Overtaking Conflicts

Program 3 is a Basic language simulation of traffic flow on a single airway (a length of 100 NM was used for the test cases below). The simulation of course allows multiple airspeeds, and computes interventions according to the formulas developed in Theorem 10. All aircraft are assumed to enter the airway along a track aligned with the airway (i.e. no turns at the entry point). The complete coding of Program 3 is given in Appendix 3 ; a brief outline of the program's structure follows:

Lines 10-350 Initialize Variables and Input Data

User inputs parameter values

- length of airway
- velocity values (Kts)
- velocity pmf
- mean traffic density (aircraft/hour)
- length of run (hours)

First aircraft is placed at airway entry point

Lines 360-620 Aircraft Arrives at Entry Point

Program determines if arriving aircraft will be overtaken by any following aircraft during its time on the airway

- if so, an overtake intervention is counted, and the arriving aircraft is diverted off the airway (i.e. removed from the data base)

The arrival time and velocity of the next arriving aircraft is calculated according to a delayed negative exponential distribution (time) and the input pmf (velocity)

Lines 630-770 Aircraft Exits Airway

As an aircraft arrives at the exit point it is removed from the data base

Lines 770-1150 Subroutines

800 Pick next velocity

Selects a velocity according to the input pmf

890 Pick distance to next entering aircraft

Selects a distance according to a delayed negative exponential distribution with parameters input by the user

950 Remove column

Deletes aircraft (column in data array) from data base

1040 Pick next critical time

Determines time of next critical event , where a critical event is an aircraft arriving at or departing from the airway

The next six tables give the results of ten simulated 8-hour runs of Program 3 in each of six test cases.

--- TEST CASE 1 ---

Velocities and relative probabilities	=	250 Kts	(.3)
		280 Kts	(.3)
		310 Kts	(.4)

Mean traffic density = 6 aircraft/hour

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 0.5788 interventions per hour

Observed intervention rates during ten simulated 8-hour periods	=	0.6157	interventions per hour
		0.4972	
		0.2494	
		0.7362	
		0.7407	
		0.2483	
		0.6129	
		0.4951	
		0.6181	
		0.3737	
Mean observed rate	=	0.5188	interventions per hour

Range about the mean = [– 52% , + 43%]

--- TEST CASE 2 ---

Velocities and relative probabilities = 300 Kts (.5)

500 Kts (.5)

Mean traffic density = 6 aircraft/hour

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 1.0138 interventions per hour

Observed intervention rates during ten simulated 8-hour periods = 0.4963 interventions per hour

0.7467

0.6198

0.6180

0.9816

0.8548

0.2493

0.7392

1.1157

0.7485

Mean observed rate = 0.7170 interventions per hour

Range about the mean = [− 65% , + 56%]

--- TEST CASE 3 ---

Velocities and relative probabilities	=	380 Kts	(.2)
		420 Kts	(.3)
		460 Kts	(.5)

Mean traffic density = 10 aircraft/hour

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 0.8587 interventions per hour

Observed intervention rates during ten simulated 8-hour periods	=	1.3281	interventions per hour
		1.1216	
		0.7443	
		0.9954	
		0.6239	
		1.6127	
		0.3746	
		0.6239	
		1.2444	
		1.1242	
Mean observed rate	=	0.9793	interventions per hour

Range about the mean = [– 36% , + 65%]

--- TEST CASE 4 ---

Velocities and relative probabilities	=	350 Kts	(.7)
		450 Kts	(.3)

Mean traffic density = 12 aircraft/hour

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 1.7797 interventions per hour

Observed intervention rates during ten simulated 8-hour periods	=	1.7495	interventions per hour
		1.7402	
		1.6136	
		2.1063	
		1.2493	
		1.2457	
		1.4952	
		1.1219	
		1.9821	
		1.3640	
Mean observed rate	=	1.5675	interventions per hour

Range about the mean = [– 28% , + 34%]

--- TEST CASE 5 ---

Velocities and relative probabilities =	300 Kts	(.2)
	320 Kts	(.3)
	350 Kts	(.5)

Mean traffic density = 9 aircraft/hour

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 0.7764 interventions per hour

Observed intervention rates during ten simulated 8-hour periods = 0.9995 interventions per hour

0.6239
1.4840
0.9997
0.2491
0.7489
1.2460
0.6230
0.2498
0.3741

Mean observed rate = 0.7598 interventions per hour

Range about the mean = [– 67% , + 95%]

--- TEST CASE 6 ---

Velocities and relative probabilities =	350 Kts	(.2)
	380 Kts	(.3)
	410 Kts	(.3)
	450 Kts	(.2)

Mean traffic density = 8 aircraft/hour

Minimum separation distance = 5 NM

Calculated intervention rate (analytic model) = 0.7121 interventions per hour

Observed intervention rates during ten simulated 8-hour periods = 1.1230 interventions per hour

0.7411
0.7323
0.6245
0.9980
0.6205
0.7466
0.6238
0.4928
0.6216

Mean observed rate = 0.7324 interventions per hour

Range about the mean = [– 33% , + 53%]

Once again the simulation data supports the validity of the analytic model. Two mean observed values were within 3% of predicted, three differed by about 10%, and one deviated by about 30%. Four mean observed values were lower than predicted by the analytic model, and two were higher. We also notice a significant variation about the observed means, on the order of $\pm$ 40% or more for all six test cases. This again is consistent with the variability observed by practicing air traffic controllers.

3.2 Variance About the Mean Intervention Rate

3.2.1 Variability in More Complex Networks

While we have seen significant variation about the mean intervention rate for single airway segments and single intersections, it does not necessarily follow that the same proportionate variability will occur in large, more complex airway networks. The key to the relationship between the variability in the total network and the variability in the individual parts is the degree to which the activity levels in the individual parts are mutually independent. For large, complex volumes of airspace (say the entire Boston Center) an assumption of significant independence may be reasonable. This is not to say that various system input parameters like traffic densities and airspeed distributions are not correlated, merely that the random variables representing actual number of controller interventions due to crossing conflicts at two spatially separated intersections may be considered largely independent once the input parameters are fixed. This independence assumption becomes even more reasonable if there is a significant amount of "randomizing traffic" : i.e. off-airway traffic which often joins/leaves airways at unpredictable points, and traffic entering/leaving the airspace from/to other altitudes.

On the other hand, smaller, simpler networks may not show as much independence in the activity levels of its constitutient parts. Consider the airway network shown in Figure 3-1. If the traffic on airway G-H is (for some reason) highly unlikely to leave G-H in this sector, and if east-west traffic generally does not turn north or south in this sector, then it would be difficult to claim independence between intersections E and F, or between airway segments A-B and C-D. Effects seen in one (i.e. short-term changes in traffic density or velocity distributions) will be reflected almost immediately in the other.

Since we see that the degree of independence between constituent parts can change significantly depending on the size, geometry, and traffic flow in an airway network, it follows that the relationship between variability in intervention rates within individual pieces (i.e. intersections, airway segments) and the total network variability can differ as well. Consider an example: a network consisting of N constituent parts, with each part having variance such that, say, 90% of its observed intervention rates will fall within 50% of the expected rate. If the actual intervention rates in the individual parts are highly independent, then the 90% interval around the mean rate for the whole network will be much less than $\pm$ 50% of the mean, since peaks in

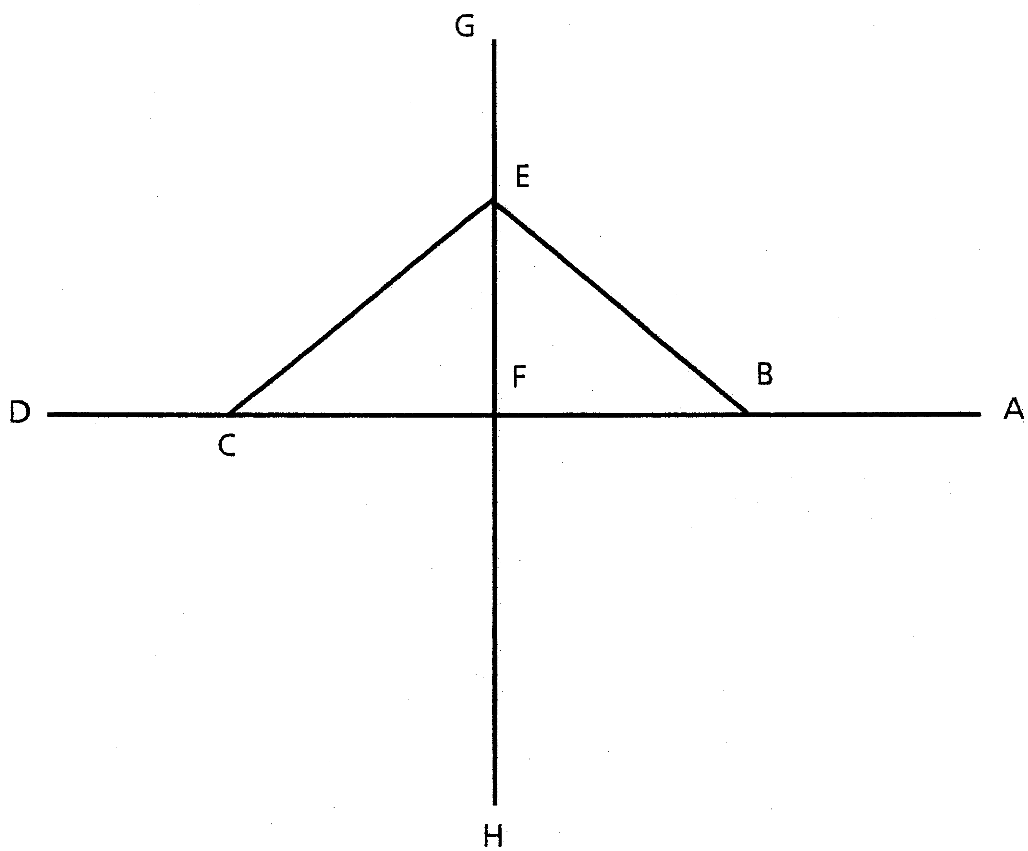

Figure 3-1

one part of the network would tend to be balanced by lulls in another. If, on the other hand, the intervention rates among the parts were highly correlated, peaks in one area would tend to correspond with peaks in another : the 90% interval for the entire network would tend to stay near $\pm$ 50% of the overall expected intervention rate.

In terms of variance we see the same kind of result. Consider two identically distributed random variables X_1 and X_2 , each having mean μ and variance σ^2 . Now

$$\mathrm{Var}[X_1 + X_2] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + 2\,\mathrm{Cov}[X_1,X_2] \quad , \tag{3-1}$$

where

$$\mathrm{Cov}[X_1,X_2] = E\,[\,(X_1 - \mu)\,(X_2 - \mu)\,] \; . \tag{3-2}$$

When X_1 and X_2 are totally dependent with correlation coefficient $\rho(X_1,X_2) = +1$, then X_1 will be large exactly when X_2 is large, and small when X_2 is small; and

$$Var[X_1 + X_2] = 4\sigma^2 . \qquad (3\text{-}3)$$

If X_1 and X_2 are independent, then $Cov[X_1,X_2] = 0$, and

$$Var[X_1 + X_2] = 2\sigma^2 . \qquad (3\text{-}4)$$

Finally, if X_1 and X_2 are again totally dependent, but their correlation coefficient, $\rho(X_1,X_2)$, is now -1 (that is, X_1 is large when X_2 is small, and vice versa), then

$$Var[X_1,X_2] = 2\sigma^2 + 2\sigma^2 = 0. \qquad (3\text{-}5)$$

This last case reflects a situation where $X_1 = -X_2$ and thus their sum is a constant zero.

If we let X_1 and X_2 represent observed intervention rates at two intersections, we get an upper bound on the variance about their expected sum when X_1 and X_2 have a correlation coefficient equal to $+1$. The variance decreases as X_1 and X_2 become more independent ($\rho(X_1,X_2)$ goes to zero) and finally reaches zero as $\rho(X_1,X_2) = -1$. This last case is unrealistic for our purposes, since it would imply the possibility of a negative number of interventions, a physically meaningless concept. Nonetheless, we see a general bound on the variance around the intervention rate at two intersections: the variance is highest when the correlation between the rates at the two intersections is high, and decreases as that correlation decreases through total independence ($\rho = 0$) towards a greater negative correlation (in limit, $\rho = -1$). We would tend to expect positive correlation in most real-world situations; that is, high traffic densities in one part of a sector will tend to occur at the same general times that high densities are found in other parts. These observations can be extended to more general situations involving sums of N random variables. using the well-known relationship

$$Var[\sum_{i=1}^{N} X_i] = \sum_{i=1}^{N} Var[X_i] + 2\sum_{j=1}^{N}\sum_{i=1}^{N} Cov[X_i,X_j] . \qquad (3-6)$$

Again we observe that the variance of the sum will tend to be high when the covariance terms are high (i.e. when $\rho \approx +1$), and will decrease as ρ tends towards -1.

Thus we see that the variance in the total intervention rate in a given sector is a function not only of the variance found in each of the sector's constituent parts, but also of the degree to which the intervention rates in the individual pieces are mutually independent.

3.2.2 An Analytic Determination of Variance

The Monte Carlo experiments just discussed have shown that a significant degree of variation may occur around the mean intervention rate. In general it is difficult to produce analytic estimates of this variation because, as we have seen, it is not always easy to determine the exact correlation between traffic densities and velocity distributions found at different locations on the network. For a simple example, however, such a calculation is possible, and will serve to confirm the variation observed in the Monte Carlo simulations.

Consider the intersection of two simple airways shown in Figure 3-2. The angle between

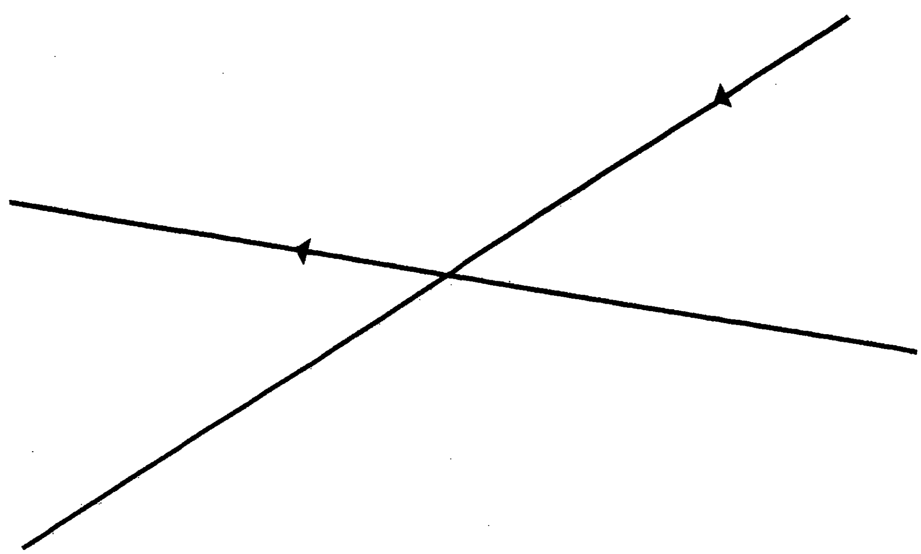

Figure 3-2

A Simple Airway Intersection

them is 30°, aircraft on both airways travel at 360 Kts, and the expected distance between successive aircraft on both airways is 60 NM. The minimum separation distance we want to maintain is 5 NM. The expected intervention rate at this intersection, calculated using the model developed in Chapter 2, is $R_c = 1.0352$. The reader should note that this situation is Test Case Six from the Monte Carlo simulations performed earlier in this chapter.

To compute the variance of X, where X is the observed number of controller interventions required during a given 8-hour period, we want

$$\sigma_x^2 = E[(X - R_c)^2] \quad .$$

Now the key to calculating σ_x^2 is to observe that the calculation of $E[(X - R_c)^2]$ is not difficult if the average distance between arriving aircraft during the 8-hour period is known beforehand (i.e. it is no longer a random variable with mean, say, 60 NM). Consider, for example, the situation pictured in Figure 3-2. If we know that the actual mean distance between aircraft during the next 8-hour period will be 40 NM on both airways (as opposed to the expected 60 NM mean separation) then we can now compute $E[(X_{40} - R_c)^2]$, where X_{40} is the number of interventions observed in an eight-hour period given that the mean separation between aircraft during that period will be exactly 40 NM. To make the calculation we simply observe that as each aircraft passes through the intersection an intervention is generated exactly when there is an aircraft upstream on the other airway within the critical miss distance. Since the velocities on both airways are constant (for this specified example), and since we know what the actual mean separation will be along each airway during the 8-hour period in question, a conflict is generated each time an aircraft passes through the intersection with (the same) probability PCON. Finally, we can compute PCON from Theorem 2 in Chapter 2. Now we have a binomial experiment with a probability of success equal to PCON, and n, the total number of samples, easily computed from the actual mean separation. Consequently, it follows immediately that

$$E[X_{40}] = n(PCON_{40}) \qquad (3-7)$$

and

$$E[(X_{40} - E[X_{40}])^2] = n(PCON_{40})(1 - PCON_{40}) \qquad (3-8)$$

From this we get

$$E[(X_{40} - R_c)^2] = E[\{(X_{40} - nPCON_{40}) + nPCON_{40} - R_c\}^2]$$

$$= \ (nPCON_{40} - R_c)^2 + \sigma^2_{X_{40}}$$

$$= \ (nPCON_{40} - R_c)^2 + n(PCON_{40})(1 - PCON_{40}) \qquad (3-9)$$

Thus we can compute $E[(X_{40} - R_c)^2]$, which is the conditional variance about R_c of the number of interventions during the 8-hour period given that the mean separation between aircraft during that period was actually 40 NM.

Using this same methodology we can compute the conditional variance of X about R_c, given any mean separation (including cases where the mean separations are not the same on both airways). Consequently, we can then compute the unconditional variance of X if we know the probability distributions of the mean separation on each airway, or equivalently, the pmf of the number of aircraft crossing the intersection during the 8-hour period.

We can compute this pmf as follows. To determine the probability that N aircraft on airway i cross the intersection during the 8-hour period, we simply compute the probability that exactly N aircraft will fall along an airway of length $L = 8V_i$, where V_i is the velocity of the aircraft on airway i. Remember, we know that the distances between the aircraft are distributed according to a delayed negative exponential distribution with mean S and delay M.

If we consider the beginning of our 8-hour period to be a random incidence into the stream of aircraft along the airway, then we know from Lemma 2 in Chapter 2 that our 8-hour period will begin with an aircraft downstream (i.e. past) the intersection at a distance $> M$ with probability $1 - M/S$. In such a case the probability that exactly N aircraft will fall within L nautical miles upstream of the intersection can be computed from the probability that exactly N aircraft will fall within a distance $L - N(M)$ nautical miles upstream if the interarrival distance is now negative exponential (not delayed) with mean $S - M$.

To see why this is so, consider what happens when exactly N aircraft with delayed negative exponential interrarivals fall within L. Since we began by assuming that our eight hour period begins when the nearest aircraft on airway i is more than M miles downstream, we can get exactly N aircraft in length L if the first N interarrival distances sum to less than L and the first $N+1$ interarrival distances sum to more than L. Now if we consider D, the interarrival distance, to be equal to M (a constant) plus E (a random variable with negative exponential pdf and mean $S - M$) then we see that exactly N aircraft fall within L precisely when the first N values of E sum to less than $L - N(M)$ and the first $N+1$ values of E sum to greater than $L - N(M)$ (See Figure 3-3.). Finally, this later probability can be computed, from the well-known result for spatial Poisson processes (e.g. Larson and Odoni [1981]) to be

$$P_1(N) = \frac{\left(\dfrac{L - N(M)}{S - M}\right)^N exp\left(-\dfrac{L - N(M)}{S - M}\right)}{N!} \quad . \tag{3-10}$$

Figure 3-3

Similar reasoning can be used when the 8-hour period begins with an aircraft downstream of the intersection on airway i at a distance $\leq$ M to give

$$P_2(N) = \frac{\left(\dfrac{L - (N + \frac{1}{2})(M)}{S - M}\right)^N exp\left(-\dfrac{L - (N + \frac{1}{2})(M)}{S - M}\right)}{N!} \quad . \tag{3-11}$$

Using Lemma 2 we compute

$$P(N) = \left(1 - \frac{M}{S}\right) P_1(N) + \left(\frac{M}{S}\right) P_2(N) \quad , \qquad (3-12)$$

the probability that exactly N aircraft will be upstream of the intersection within a distance L, where the interarrival distances are DNE with delay M and mean S − M.

Now that we can compute P(N) we are in a position to compute $\sigma_X{}^2$, the variance of the number of interventions in an 8-hour period. To wit:

$$\sigma_X^2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{\#1}(i)\, P_{\#2}(j)\, E[(X_{ij} - R_c)^2] \quad , \qquad (3-13)$$

where

$P_{\#1}(i)$ ≡ probability that i aircraft on Airway #1 will cross the intersection during a random 8-hour period

$P_{\#2}(j)$ ≡ probability that j aircraft on Airway #2 will cross the intersection during a random 8-hour period

R_c ≡ the expected intervention rate at the intersection

X_{ij} ≡ a random variable representing the number of interventions at the intersection given i aircraft pass by on Airway #1 and j aircraft pass by on Airway #2.

A simple computer inplementation of equation 3-13 can be written to calculate the predicted variance about the mean intervention rate at a simple intersection of the type used in the simulations in this chapter (see Program 5 in the Appendix). In Table 3-1 we compare the sample variance

$$S^2 = \frac{\sum_{i=1}^{n} (X - m)^2}{n-1} \qquad (3-14)$$

to the variance predicted by the analytic model just developed (equation 3-13), where X is the number of interventions observed in an eight-hour period, and m is the sample mean.

Test Case	Sample Variance	Predicted Variance
3	12.16	37.78
5	13.79	9.49
6	4.30	8.36

Table 3-1

We can indicate the degree to which the results in Table 3-1 support the analytic estimate of intervention rate variance by constructing 95% confidence intervals about the sample variances. These confidence intervals, presented in Table 3-2, support the validity of the analytic model, since the predicted variances fall within the 95% confidence intervals in each test case. The reader should note that the confidence intervals presented in Table 3-2 are based on

Test Case	Predicted Mean	Predicted Variance	95% Confidence Interval Around Sample Variance
3	3.52	37.78	[5.75 , 40.53]
5	1.15	9.49	[6.52 , 45.96]
6	1.04	8.36	[2.03 , 14.33]

Table 3-2

the assumption that the number of interventions observed in an eight-hour period is normally distributed about the mean, a reasonable assumption in light of the Central Limit Theorem.

An analytic prediction of the variance about the mean for the simple overtaking situations simulated in this chapter can be constructed in a manner analagous to the one used for the simple intersection. The number of iterations required to calculate the conditional probabilities will be significantly larger, however, since it will be necessary to condition not only on the actual traffic density, but also on the various ways that each density might be partitioned among the various velocity classes (e.g. 26 aircraft per hour with 1 of velocity A and 25 of velocity B; 26 aircraft per hour with 2 of velocity A and 24 of velocity B; . . .).

3.3 Validation Using Real-World Data

We have seen that the Monte Carlo simulations described in this chapter are consistent with the predictions of the analytic model developed in Chapter 2. One would suppose that the next logical step would be to compare such predictions with intervention rates gleaned from observations of actual controllers at work in operational en route sectors.

Unfortunately, it is not easy to make such a comparison, because of difficulties surrounding the gathering of data on actual intervention rates. In the first place, it is not easy to define exactly when an intervention has occurred. As an example, consider a case where a controller receives a request for a new route of flight from one of the aircraft in his sector. Before granting such a request, the controller will look ahead to insure the requested change will not generate a possible conflict along the new route. If such a conflict potential exists, the controller may deny the request, or he may delay granting approval until the potential for conflict no longer exists. If he chooses the later, is this action to count as an intervention in response to an impending conflict? Conversely, a controller may direct an early descent out of the high altitude structure for an aircraft due to begin transition to the landing phase, if such a descent will clear that aircraft from a developing conflict at its current cruise altitude. Is this early descent an intervention, and if so, how is it to be recognized as such?

Another problem with real-world data on controller activities is that judgments about developing conflicts can be very subjective. Since controllers are subject to disciplinary action if they allow a minimum separation standard violation to occur in their sector, most controllers give themselves a "pad" of up to several miles beyond the minimum separation. Any developing situation which threatens to allow separation to decrease to the "padded" minimum (i.e. the actual minimum plus the controller's own personal safety factor) will trigger an intervention. Further, the size of the pad may be a function of several factors, including the identity of the controller, his experience level and workload, and the geometry of the developing conflict (Dunlay [1974], [1975]).

Another problem with using observed intervention rates is the fact that interventions may be separated in space and time from the conflict which triggers them. As an example, controllers may use separation standards well in excess of the minimum for streams of aircraft at high altitude that are scheduled to land at the same airport. This additional separation is used to smooth the flow of these aircraft into the low altitude and approach to landing phases of flight, which may not occur until well after the aircraft have left their current controller's sector. Thus the controller's actions which serve to maintain the increased separation are in fact interventions intended to avoid conflicts that may develop in the future. It would be difficult to recognize and properly allocate such interventions from the raw data on controller activities.

Due to these difficulties surrounding the definition and recognition of interventions caused by impending conflicts, we have not attempted to validate the model developed in Chapter 2 by using such data. These difficulties point out the fact that the analytic model predicts one portion of the controller's workload - interventions due to developing conflicts. This portion is well integrated with his other traffic monitoring and communication duties and is difficult to isolate in actual practice.

3.4 Conclusions

The results of this chapter support the analytic model developed in Chapter 2 as a good predictor of expected controller intervention rates due to crossing and overtaking conflicts. We have seen, however, that variation about the expected rate must also be taken into account, since deviations from the expected value over a typical 8-hour shift can be $\pm$ 40% or more (and even greater deviations are likely during shorter peak periods).

A complete analysis of controller intervention rates in a given air traffic control sector will require the use of both analytic and simulation models. The analytic model will give a rapid calculation of the mean intervention rate, and will show how this rate changes as a function of sector geometry, traffic density, and the distribution of velocities. The Monte Carlo simulation, on the other hand, will provide an indication of the variability to be anticipated about that expected rate.

Chapter 4

Sensitivity Analysis

4.1 Introduction

In this section we will examine the sensitivity of the model to some key assumptions. We will consider the effects of the following:

1. The assumption that we know the precise position and velocity of each aircraft.

2. The assumption that airway traffic is always exactly on centerline.

3. The assumption that the interarrival spacing between successive aircraft along an airway is distributed according to a delayed negative exponential distribution.

4. The assumption that a controller will always respond in a specific way to a given conflict situation.

5. The assumption that the system is in steady state.

6. The assumption that the DNE interarrival distribution can be decomposed

4.2 Imperfect Knowledge of Position and Velocity

What effect do we have on the results of the model if we admit that we do not have perfect knowledge of the precise position and velocity of each aircraft? Uncertainty concerning aircraft position turns out to be relatively unimportant for the purposes of this model. Since we are trying to predict controller intervention rates, and since controllers intervene on the basis of the information presented to them (ie. the picture on their radar scope) , it will not affect the intervention rate if the scope presentation is slightly in error. To be sure, any error in the radar presentation will have a definite effect on flight safety, but that question is beyond the scope of this model.

Possible error in our knowledge of aircraft velocity is another matter. In effect, we are asking how well we know the distribution of airspeeds along an airway, and of course changes in these distributions can have significant effects upon the results of the model. Two sources of uncertainty are of interest. The first concerns uncertainty about the variation around a known mean velocity, while the second involves errors in estimating the mean itself. We will look at both types of uncertainty, first for overtaking conflicts.

From the data in Table 4-1 we can make the following observations:

1. Uncertainty about airspeed distributions can have a significant effect on predicted R_0. For velocities typical of the high altitude structure, errors on the order of 10% can cause R_0 to deviate from 15-100%.

2. Both errors in variability about a known mean and errors in the mean itself can significantly change R_0.

3. If actual velocities are all higher than expected, the actual R_0 will decrease (since the aircraft spend less time on the airway). A uniform decrease in airspeeds or a mixture of increases and decreases, tend to increase R_0.

Notice that observation 3 addresses the effect of changing headwinds/tailwinds (to the extent that there is no compensation made by the aircrews or the controller). An increased headwind which slows all traffic on the airway will tend to increase R_0 , because the aircraft spend a longer time on the airway, the aircraft tend to be closer together (for the same traffic density in aircraft/hour), and their relative closure rates are unchanged. An increased tailwind, of course, will have the opposite effect.

Length of Airway (NM)	Minimum Separation (NM)	Traffic Density (A/C per hr)	Velocities (Kts) and Probabilities	Predicted Overtake Intervention Rate (R_o)
100	5	12	350 (.5) 450 (.5)	2.011
100	5	12	315 (.5) 415 (.5)	2.342
100	5	12	385 (.5) 485 (.5)	1.740
100	5	12	315 (.5) 485 (.5)	3.056
100	5	12	315 (1/6) 350 (1/6) 385 (1/6) 410 (1/6) 450 (1/6) 490 (1/6)	2.480
100	5	12	420 (1/3) 450 (1/3) 480 (1/3)	0.920
100	5	12	380 (1/3) 410 (1/3) 440 (1/3)	1.095
100	5	12	460 (1/3) 490 (1/3) 520 (1/3)	0.782
100	5	12	380 (1/3) 450 (1/3) 520 (1/3)	1.958
100	5	12	380 (1/6) 410 (1/6) 440 (1/6) 460 (1/6) 490 (1/6) 520 (1/6)	1.690

Table 4-1
Effect of Velocity Uncertainty on Overtaking Intervention Rate

Now lets consider the effect of velocity uncertainties on crossing conflicts.

Angle Between Airways (deg)	Traffic Density (A/C per hr)	Minimum Separation (NM)	Vel. and Probabilities		Crossing Intervention Rate (R_c)
			A/W 1	A/W 2	
45	12	5	450 (1.0)	450 (1.0)	3.462
45	12	5	405 (1.0)	405 (1.0)	3.846
45	12	5	495 (1.0)	495 (1.0)	3,147
45	12	5	405 (1.0)	495 (1.0)	3.594
45	12	5	405 (1/3) 450 (1/3) 495 (1/3)	405 (1/3) 450 (1/3) 495 (1/3)	3.520

Table 4-2.1
Effects of Velocity Uncertainty on Crossing Intervention Rate

Angle Between Airways (deg)	Traffic Density (A/C per hr)	Minimum Separation (NM)	Velocities and Probabilities		Crossing Intervention Rate (R_c)
			A/W 1	A/W 2	
45	12	5	420 (1.0)	460 (1.0)	3.568
45	12	5	380 (1.0)	415 (1.0)	3.948
45	12	5	460 (1.0)	505 (1.0)	3.256
45	12	5	380 (1.0)	505 (1.0)	3.787
45	12	5	380 (1/3) 420 (1/3) 460 (1/3)	415 (1/3) 460 (1/3) 505 (1/3)	3.624

Table 4-2.2
Effects of Velocity Uncertainty on Crossing Intervention Rate

Observations:

1. Ten percent changes in velocity (both around the mean and changes to the mean itself) have some effect on controller interventions due to crossing conflicts. For typical en route velocities the change in R_c is on the order of 0-10%.

2. Uniformly increasing velocities at an intersection tend to decrease the intervention rate (assuming traffic density is held constant), since distances between aircraft then tend to be greater. Uniformly decreasing velocities has the opposite effect.

3. Mixed changes in velocities - some increases and some decreases - have relatively little effect on R_c. Thus, uncertainty about the variance around a known mean will have small effect on the intervention rate due to crossing conflicts.

In summary, changes in velocity distributions can have a significant effect on controller intervention rates. The effect is particularly noticeable with regards to overtaking conflicts (as much as 100% change in R_0 as a result of a 10% increased velocity in fast aircraft and a 10% decrease in slow aircraft) , while the effect on R_c is generally much less apparent.

4.3 Aircraft Cross-Track Deviation

In Chapter 2 we assumed that aircraft never deviate from the airway centerline to which they are assigned. Of course, any controller will tell you that this is hardly the case: the deviations are usually noticeable (1-2 miles) and occasionally quite significant ($>$ 10 miles). In the case of overtaking conflicts, a large cross-track variation might theoretically reduce controller interventions, since overtaking and overtaken traffic might happen to deviate to opposite sides of the airway and pass one another without requiring a controller intervention. In actual practice there is probably not a significant underestimate of workload due to overtakes, because the wide cross-track deviations will still make demands upon the controllers' attention: if nothing else, he must be prepared with a plan of action if one or both aircraft begin to return to their assigned airway as they are passing.

Cross-track deviations may have a more significant impact on the intervention rate due to crossing conflicts at an intersection, so we need to look at that situation in more detail. The model developed in Chapter 2 was not designed to facilitate an analysis of the way intervention rates change with incremental changes in intersection geometry. A more useful approach will be one found in Endoh [1982], where he considers the conflict rate to be proportional to the overlap area between the two airways. By overlap area we mean the intersection of the locus of points within 5 NM of each airway centerline (see Figure 4-1).

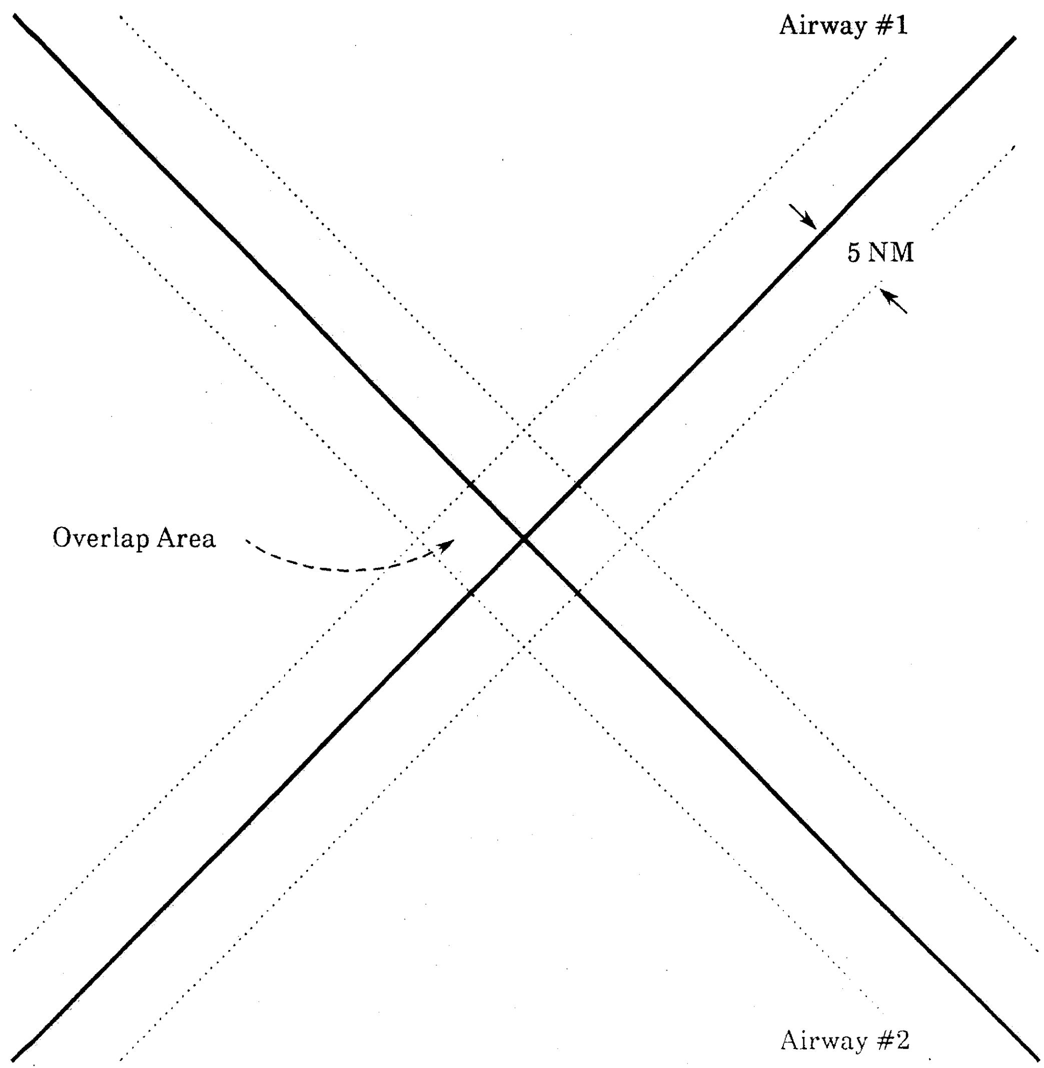

Figure 4-1

Overlap Area

Endoh showed that for a given traffic density, the crossing conflict rate at the intersection of two straight airways is proportional to the overlap area. We will extend this concept to airways which may change ground track at the intersection, and use the variation in conflict area as a measure of the variation in conflict rate.

Consider first the intersection of two straight airways (i.e. airways that don't change ground track at the intersection) as shown in Figure 4-1 . If we represent cross-track deviation on Airway #1 by moving Airway #1's centerline left or right, then we see that cross-track deviations will not change the overlap area (and thus the conflict rate) at all. Therefore intervention rates due to crossing conflicts at the intersection of straight airways (where we assume aircraft do not change airways at the intersection) are not affected by cross-track deviations.

Now lets consider the intersection of a straight airway with one that changes ground track at the intersection , as in Figure 4-2. Moving Airway #1's centerline left or right will change the overlap area, and thus the conflict rate. Deviating to the left of track on Airway #1 will increase the overlap area (since the upstream protion of Airway #2 makes a more acute angle with Airway #1) , while deviating to the right of track will decrease it. As an example of the order of magnitude of the change, assume the following:

Airway #1 ground track :		045°
Airway #2 ground track		
	inbound:	090°
	outbound:	135°

A little geometry shows that when there is no cross-track deviation on Airway #1, the overlap area measures 122.85 NM^2. When we move the centerline of Airway #1 5 NM to the right (to simulate a 5 NM deviation right of track), the overlap area is reduced to $10 \times 10 = 100$ NM^2, while a deviation left of track gives an overlap area of $10 \times 10 \times \sec(\theta) = 141.42$ NM^2. Here we define $\theta \equiv 90 - \alpha$, where α is the angle between the airways. Thus the overlap varies about 20% as we move 5 NM left or right of track.

As α approaches 90° the scenario tends toward the intersection of two straight airways and the change in overlap area becomes small (at $\alpha = 90°$ the change is zero, as we saw above). For $\alpha = 45°$ the change in overlap area is linear over most of the $\pm$ 5 NM cross-track deviation range - the only exception being a non-linear (actually quadratic) portion as the cross-track deviation gets close to 5 NM right of track. Finally, as α approaches 0° the non-linear portion approaches zero. Thus it is reasonable to approximate the change in overlap area as being linear with the cross-track deviation for small α , and as being small as α approaches 90°. If we make

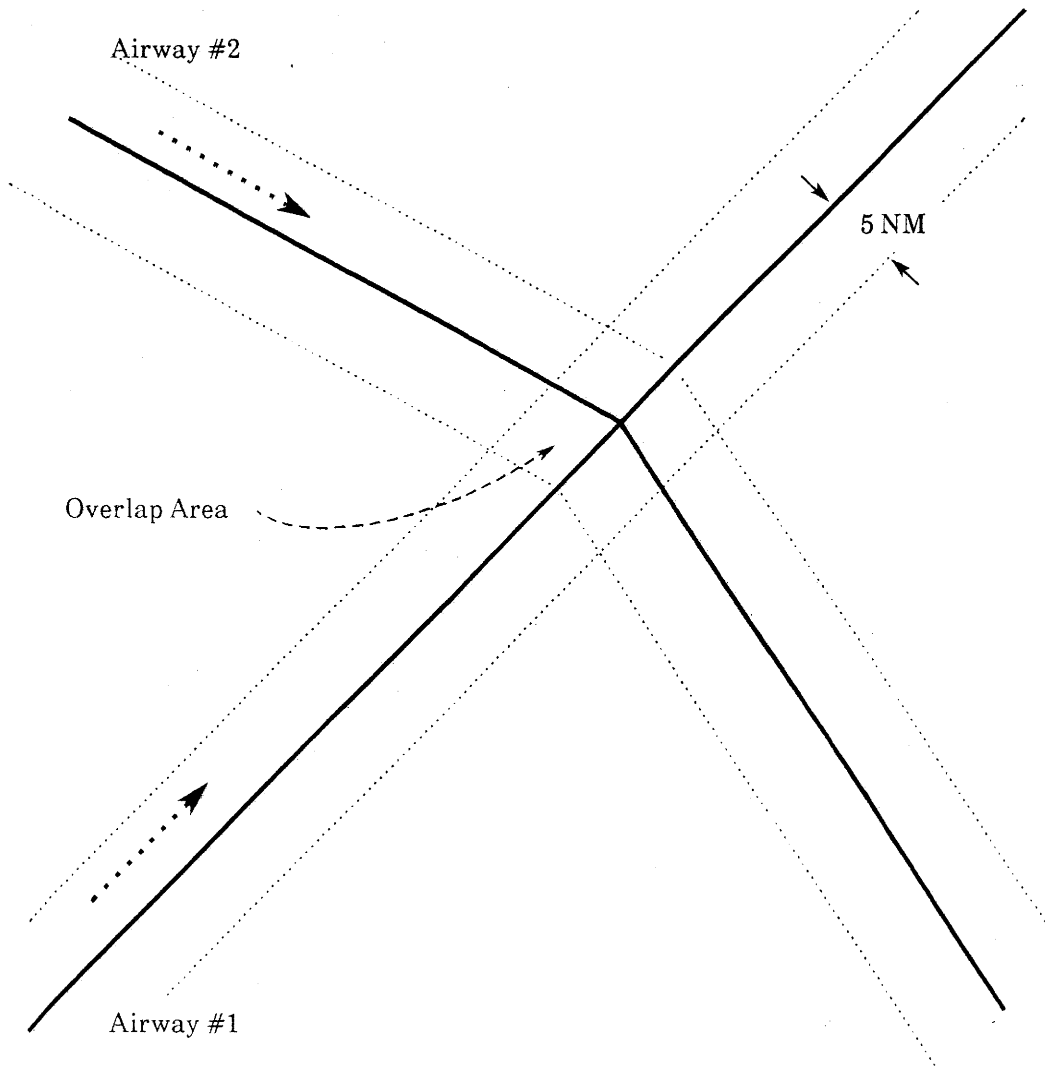

Figure 4-2

Overlap Area When One Airway Changes Groundtrack

the reasonable assumption that cross-track deviations are distributed symmetrically about the centerline, then the expected conflict area, and thus the intervention rate, becomes largely insensitive to cross-track deviations.

A similar line of reasoning can be used to deal with more complex situations involving two airways, each of which change ground track at the intersection. When the change in overlap area is not linear as a function of deviation, it tends to be small, so that we can again conclude that the intervention rate is relatively insensitive to cross-track deviations.

4.4 Interarrival Distribution

At the beginning of Chapter 2 we discussed the distribution of interarrival distances at entry points into the sector. We showed why the delayed negative exponential (DNE) distribution would more realistically reflect the efforts of controllers in neighboring sectors to maintain minimum separation.

Now we will consider the impact of this assumption on our results : specifically, we will compare intervention rates computed using a DNE interarrival distribution with what we would compute using a standard negative exponential distribution. Of course, we will hold traffic density (aircraft arriving per hour) constant.

First we will consider the calculation of R_c , the intervention rate due to crossing conflicts. In the development of the model in Chapter 2 the interarrival distribution came into play at only one point: during the proofs of Lemma 2 and Theorem 2 which lead to an expression for $PCON_1$, the probability that an aircraft on airway #1 will be within its critical miss distance, CM, of the next aircraft inbound on airway #2. From Theorem 2 we have

$$PCON_1 = 1 - \left(\frac{S_2 - M}{S_2} \right) exp\left(\frac{M - CM}{S_2 - M} \right) , \qquad (4-1)$$

where

S_2 ≡ expected separation between aircraft on Airway #2

M ≡ minimum allowed separation

C ≡ a factor computed according to Theorem 1 (i.e. CM is the critical miss distance)

Now the final crossing intervention rate R_c varies linearly with each of the various PCON (all else being equal) so to judge the effect of a different interarrival distribution we need only observe its effect on the computation of PCON.

If we assume a negative exponential distribution on interarrival distances, then the computation of PCON is simple. We are merely interested in determining the probability that the next arriving aircraft on Airway #2 will be less than MC away. Thus

$$PCON = \int_0^{CM} \frac{1}{S_2} exp\left(- \frac{x}{S_2} \right) dx$$

$$= 1 - exp\left(- \frac{CM}{S_2} \right). \qquad (4-2)$$

A comparison of PCON's computed with both DNE and NE distributions follows in Tables 4-3.

Angle Between Airways (deg)	Mean Arrival Separation (NM)		Velocity (Kts)		PCON		Percent Difference Between PCONs
	A/W 1	A/W 2	A/W 1	A/W 2	DNE	NE	
30	20	20	360	360	.2588	.2280	11.9
60	20	20	360	360	.2877	.2507	12.8
90	20	20	360	360	.3467	.2978	14.1
120	20	20	360	360	.4626	.3935	14.9
150	20	20	360	360	.7113	.6194	12.9
30	60	60	360	360	.0863	.0827	4.1
60	60	60	360	360	.0961	.0917	4.6
90	60	60	360	360	.1172	.1112	5.1
120	60	60	360	360	.1630	.1535	5.8
150	60	60	360	360	.2934	.2752	6.2

Table 4-3.1
Effect of Interarrival Distribution on PCON

Angle Between Airways (deg)	Mean Arrival Separation (NM)		Velocity (Kts)		PCON		Percent Difference Between PCONs
	A/W 1	A/W 2	A/W 1	A/W 2	DNE	NE	
60	10	10	360	360	.5717	.4386	23.3
60	6	6	360	360	.9231	.6180	33.1
30	60	60	300	540	.1720	.1619	5.8
60	60	60	300	540	.1479	.1396	5.7
90	60	60	300	540	.1675	.1577	5.9
120	60	60	300	540	.2244	.2106	6.1
150	60	60	300	540	.3869	.3637	6.0

Table 4-3.2
Effect of Interarrival Distribution on PCON

From the data in Tables 4-3 we can make the following observations:

1. The probability of conflict computed using a delayed negative exponential interarrival distribution is <u>higher</u> than the PCON predicted using a standard (non-delayed) negative exponential distribution.

2. The difference between the two PCON values increases as traffic becomes more dense. The difference does not vary significantly with the angle between the airways.

3. At traffic densities typical of high altitude airways, the values differ by about 5-10%.

An intuitive explanation for the fact that the NE PCON is the lower of the two is that a non-delayed interarrival distribution occasionally generates two aircraft in very close trail. These aircraft will be generating interventions almost as if they were a single aircraft, thus reducing the overall intervention rate.

Now lets look at the effect of a NE interarrival distribution upon R_0, the intervention rate due to overtaking conflicts. In Theorem 10 we have computed $P_{NO}(v_1, v_2)$, the probability that an aircraft of velocity v_1 will not be overtaken by an aircraft of velocity v_2, to be

$$P_{NO}(v_1, v_2) = \begin{cases} 1 & \text{if } v_1 \geq v_2 \\ exp\left(\dfrac{(v_1 - v_2)L}{v_1(S_{v_2} - M)}\right) & \text{if } v_1 < v_2 \end{cases} \qquad (4-3)$$

where

L $\equiv$ the length of the airway

M $\equiv$ the minimum separation distance

S_{v_2} $\equiv$ the expected distance between aircraft of velocity v_2

Theorem 10 assumed a delayed negative exponential interarrival distribution. If, instead, we assume a standard negative exponential interarrival distribution, then, following Theorem 10, we can compute

$$P_{NO} = \int_{d(v_1, v_2)}^{\infty} \frac{1}{S_{v_2}} exp\left(-\frac{x}{S_{v_2}}\right) dx$$

$$= \int_{M + \frac{(v_2 - v_1) L}{v_1}}^{\infty} \frac{1}{S_{v_2}} exp\left(- \frac{x}{S_{v_2}} \right) dx \; . \qquad (4-4)$$

Thus

$$P_{NO} \quad \begin{cases} = exp\left(\dfrac{-Mv_1 - (v_2 - v_1) L}{v_1 S_{v_2}} \right) & \text{if } v_1 < v_2 \\ = 1 & \text{if } v_1 \geq v_2 \end{cases} \qquad (4\text{-}5)$$

Here $d(v_1, v_2) = M + (v_2 - v_1) L / v_1$ is the minimum separation between an aircraft of velocity v_1 at the airway entry point and a following aircraft of velocity v_2, such that the later won't overtake the former before the former arrives at the departure end of the airway. Of course, if $v_1 > v_2$ then $d(v_1, v_2) = 0$ and $P_{NO} = 1$.

Table 4-4 shows R_o, the controller intervention rate due to overtakes along a single 100 NM airway, with R_o computed using both DNE and NE interarrival distributions.

Airway Length (NM)	Min. Separat. Dist (NM)	Traffic Density (A/C per hour)	Velocities (Kts) and Probabilities	Overtaking Intervention Rate DNE	Overtaking Intervention Rate NE	Percent Difference Between R_o's
100	5	6	350 (.5) 450 (.5)	.5365	.8006	49.2
100	5	6	350 (.4) 400 (.3) 450 (.3)	.4897	.7355	50.2
100	5	6	320 (.2) 350 (.3) 380 (.3) 410 (.2)	.4416	.6957	57.5
100	5	15	380 (.4) 410 (.2) 460 (.4)	2.257	3.476	54.0
100	5	40	380 (.4) 410 (.2) 460 (.4)	13.65	18.49	35.4

Table 4-4
The Effect of Interarrival Distribution on R_o

Not unexpectedly, the use of NE vs DNE makes a bigger difference when computing overtaking intervention rates: the value for R_0 is significantly higher when using NE. For values typical of the en route air traffic system, a difference of 50% or more is not unusual. The explanation for this result should be clear : a negative exponential distrtibution allows aircraft to enter the sector with separations less than M (the minimum acceptable), generating immediate overtaking conflicts.

Thus the computation of the overall intervention rate within a given sector <u>is</u> sensitive to the interarrival distribution. The difference may be as high as 50% , depending on the ratio of overtaking to crossing conflicts.

4.5 Controller Response to Impending Conflicts

In the course of developing this model we had to make certain assumptions about the way controllers respond to conflict situations. We assumed, for example, that controllers will always divert the slower aircraft in an overtaking situation, and the aircraft closer to the intersection in a crossing conflict. Further, we assumed that the controller would divert an aircraft (by giving heading, airspeed, or altitude changes) so that the diverted aircraft would be involved in no further conflicts in that sector. The first assumption probably models occasionally inefficient behavior (i.e. it counts too many interventions) because there are certainly cases where diverting the slower aircraft (or the one further from the intersection) may preclude the development of another conflict a few minutes later. On the other hand, the second assumption is probably optimistic, since it is not always possible to clear an aircraft around all developing conflicts with a single intervention. Both types of errors become more significant as traffic densities increase. At densities typical of the en route structure the impact of such errors will be small since the probability of a single aircraft being involved in multiple conflicts is small.

We should note that the definition of "intervention" has remained necessarily vague : when does a series of instructions (and radio calls) become more than a single intervention? Experienced controllers will admit to no fixed guidelines for conflict resolution: each situation may have unique characteristics requiring an unusual response. Consequently the model developed herein can lay no claims to predicting optimal or even probable controller behavior in every situation. For predicting controller average workloads over extended periods of time under reasonable traffic loads, the assumptions made above appear to be appropriate.

4.6 Assuming Steady State

In the development of the analytic model in Chapter 2, as well as in the Monte Carlo simulations in Chapter 3, we used traffic density values reflecting the average expected value over the entire period of interest. Thus we computed the expected intervention rate at an intersection over an eight-hour period using the average expected traffic density over that same eight-hour period. We did not stop to consider the effects of variations in mean rate during the period; for example, we did not compare the intervention rate at an intersection having flow rate S over eight hours to the rate at the same intersection having flow rate 2S during hours 0 - 4 , and zero flow during hours 5 - 8. In this section we will examine the sensitivity of the model to such variations in mean flow rate. We will determine whether it is necessary to subdivide our periods of interest to more accurately reflect the effects of changing mean traffic densities on the overall intervention rate.

In Tables 4-5 we see the results of calculating the crossing intervention rate, R_c, at the

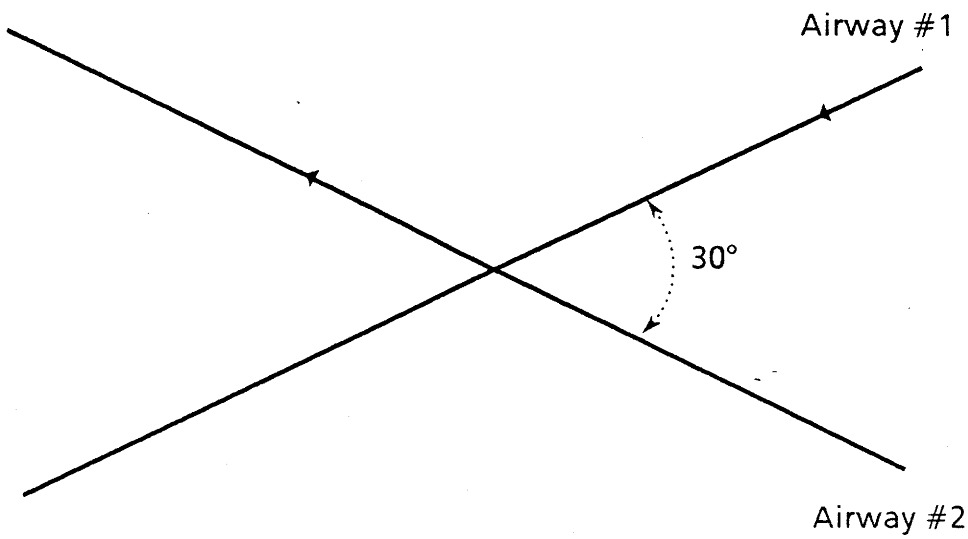

Figure 4-3

simple intersection shown in Figure 4-3. For all of the calculations in Tables 4-5 we assumed that the minimum permissible separation was 5 NM , and that 50% of the aircraft on each airway had a 420 kt velocity, and the other 50% , 460 kt. From the data in Tables 4-5 we see that our steady state assumption may or may not make a difference in the predicted intervention rate, depending how the mean traffic density varies over time.

Traffic Density Airway #1 (Aircraft/hr)	Traffic Density Airway #2 (Aircraft/hr)	Crossing Intervention Rate (per hour)
0	20	0
1	19	.451
2	18	.855
5	15	1.78
10	10	2.37
0	40	0
1	39	.926
2	18	1.8
5	35	4.15
10	30	7.12
15	25	8.90
20	20	9.5

Table 4-5.1

Referring to Figure 4-3, consider the case where the mean traffic density on one airway ,say Airway #1, is constant during the period in question, while the mean density on the other is in transition (see Tables 4-5.2 and 4-5.3). Now we find that if the long term average density on Airway #2 does not change, the mean intervention rate $\bar{R}_c$ will not change either. To see this, set the density on Airway #1 at 20 aircraft/hour and assume the long term (eg. eight-hour) average density on Airway #2 is also 20 aircraft/hour. Then over an eight-hour period we will expect

$$8 \times R_c = 8 \times 9.50 = 76$$

total crossing interventions. Compare this to a situation where the density on Airway #2 is 30 aircraft/hour for four hours and 10 aircraft/hour for the remaining four hours (for an average density of 20 over the eight-hour period) . Then we would expect the total number of crossing

Traffic Density Airway #1 (Aircraft/hr)	Traffic Density Airway #2 (Aircraft/hr)	Crossing Intervention Rate (per hour)
20	0	0
20	1	.475
20	2	.950
20	5	2.37
20	10	4.75
20	15	7.12
20	20	9.5
20	25	11.87
20	30	14.24
20	35	16.62
20	40	18.99
5	5	.594
20	20	9.50
40	40	37.98

Table 4-5.2

interventions to be

$$4 \times 14.24 \ + \ 4 \times 4.75 \ = \ 75.96 \, .$$

The reader can convince himself from the data in Tables 4-5 that this result appears to hold for other traffic densities as well, as long as the mean density on one airway is constant, while the mean density on the other airway, although transient, averages to some (perhaps different) constant over the period in question.

As another example, consider a situation in which the average density over eight hours on each airway is, say, 20 aircraft/hour. In this case, assuming steady state, we again calculate $R_c = 9.50$. But if the mean density on both airways is in transition, and if the two means are positively correlated, then the expected intervention rate will actually be greater than the steady

Traffic Density Airway #1 (Aircraft/hr)	Traffic Density Airway #2 (Aircraft/hr)	Crossing Intervention Rate (per hour)
40	0	0
40	5	4.75
40	10	9.5
40	15	14.24
40	20	18.99
40	25	23.74
40	30	28.49
40	35	33.23
40	40	37.98
80	0	0
80	5	9.49
80	10	18.99
80	20	37.97
80	30	56.95
80	40	75.93
80	50	94.91
80	60	113.88
80	70	132.85
80	80	151.81

Table 5-5.3

state value of 9.50 intervention per hour. If, for example, the density on both airways is alternatively 40 and 0 (still averaging to 20 aircraft per hour over the eight hour period), with a maximum positive correlation of $\rho = +1$, then the expected number of interventions will be

$$4 \times 37.98 + 4 \times 0 = 151.92,$$

much greater than the steady state prediction of

$$8 \times 9.5 = 76$$

interventions during the eight hour period. Once again, the values for R_c are taken from Table 4-5. If we modify this example, so that the traffic rates on the two airways have a very high negative correlation (i.e. peak flow on one airway corresponds to zero flow on the other) we get a significantly different result. If we calculate R_c over eight hours, using a mean density of 20 aircraft/hour for both airways, we would again get $R_c = 9.50$ interventions per hour. If, however, we break the eight-hour period into smaller - say one hour - segments, so that one or the other airway would always have zero flow, we calculate $R_c = 0$! Thus the effects of transient behavior of the flow rate on the calculation of R_c depends on how the flow rates are correlated between airways. If the flows are independent (i.e. one can be considered constant with respect to the other), we see little effect, but if positive or negative correlation exists we can have significant error in our predicted intervention rates if we assume steady state where transient behavior exists.

Analytic confirmation of these observations can be derived from an observation made in Chapter 2. We saw there that Endoh's predicted conflict rates at intersections of airways with constant groundtracks agree to within 1% of the intervention rates predicted by the model developed in this thesis, as long as traffic densities are typical of those actually found in the en route air traffic system. Since Endoh's conflict rate, E, for this sort of intersection is given by

$$E = \frac{2M(v_1^2 + v_2^2 - 2v_1 v_2 \cos\alpha)^{\frac{1}{2}}}{S_1 S_2 \sin\alpha} \qquad (4-6)$$

it follows that

$$E \propto \lambda_1 \lambda_2 \qquad (4-7)$$

where λ_1 and λ_2 represent the traffic densities on airways #1 and #2, respectively. Thus

$$R_c \approx E \propto \lambda_1 \lambda_2 \qquad (4-8)$$

and so, if λ_1 is constant,

$$R_c \propto \lambda_1\lambda_2 = \lambda_1\left(\sum_{i=1}^{N} a_i\lambda_2^{(i)}\right) \qquad (4-9)$$

as long as

$$\sum_{i=1}^{N} a_i\lambda_2^{(i)} = \lambda_2 \ . \qquad (4-10)$$

This confirms our observation that R_c does not change as long as one of the two airways in question does not exhibit any transient flow rates. This result also applies if both airways have transient flow rates, but the rates are independent of one another, since

$$E[\lambda_1\lambda_2] = E[\lambda_1]E[\lambda_2] \qquad (4-11)$$

for independent λ_i's.

In the more general case, where the λ_i's are not independent, we have by definition of covariance

$$E[\lambda_1\lambda_2] = E[\lambda_1]E[\lambda_2] + Cov[\lambda_1, \lambda_2] \ . \qquad (4-12)$$

Since $R_c \propto \lambda_1 \lambda_2$, it follows that $R_c \propto E[\lambda_1 \lambda_2]$ when λ_1 and λ_2 vary over time. Thus R_c will be maximaized when $\rho_{\lambda_1,\lambda_2} = +1$ (this maximizes $Cov[\lambda_1, \lambda_2]$), and minimum when $\rho_{\lambda_1,\lambda_2}$ is minimum. These results are consistent with the observations drawn from Tables 4-5.

Similar results can be shown for R_o, the overtaking conflict rate. When the mean densities for each velocity group are in transition and negatively correlated (i.e. the airway tends to be saturated at any given time with aircraft of the same velocity), then the predicted R_o, based on an assumption of steady state, will err on the high side. Conversely, if the transitory mean densities tend to be positively correlated, so that busy periods will involve aircraft of all velocities, the steady state predicted R_c will tend to be too low.

In conclusion, we see that the model developed in Chapter 2 can be sensitive to the assumption that traffic densities are in steady state. In some rather extreme examples, the steady state predicted values for R_c and R_o may be noticeably in error if the system paramenters are actually in transition. In applications where clear peak periods are observed, it would be advisable to compute total sector intervention rates over periods small enough to isolate noticeable peak or slow traffic periods, particularly when - as is often the case - these transient flow rates correlate with the transient behavior of traffic in other areas of the sector. This sensitivity to long-term aggregation of traffic flow rates is consistent with pilot (and controller) observations that intervention rates during peak traffic periods can be significantly greater than average rates predicted for longer periods of several hours or more.

4.7 Decomposition of the DNE Interarrival Distribution

In the development of the complete two-dimensional model in Chapter 2 we made use of a very important assumption about the behavior of the delayed negative exponential distribution (DNE). The reader will remember it was assumed that arrivals along any airway were separated by a distance which was a random variable distributed according to a DNE. The delay was equal to the minimum horizontal separation standard, and the mean separation was set to conform to the observed traffic density along that airway. When the simple intersection model (Section 2.1.1) was extended to more complex situations involving multiple airspeeds and aircraft airway changes at the intersection, we assumed that the flow of each airspeed and pseudo-airway group was also governed by a DNE on the interarrival distances.

An example will clarify this assumption. Let the traffic on Airway #1 have an expected interarrival distance of 30 NM. Further, let the aircraft on Airway #1 be equally divided into two groups, one with 400 kt velocity and the other with 450 kt. We assume, as mentioned above, that the interarrival distance between any two successive aircraft on Airway #1 is distributed according to a DNE with a delay of M (= the minimum separation standard, usually 5 NM) and an expected value of 30 NM. In addition, we also assume that the distance between two successive aircraft <u>of the same airspeed</u> is also distributed according to a DNE with delay M and mean $30/.5 = 60$ NM. We used this same "decomposition" assumption when we looked at traffic flow along pseudo-airways in Section 2.1.5, and when we computed overtake interventions in Section 2.2 .

For general interarrival distributions this decomposition assumption is not true. Consider, for example, a stream of arrivals that are separated by a constant distance D. If we divide the arrivals into N randomly assigned, equally probable groups, then the interarrival distance between successive arrivals in the same group is certainly not a constant $N \times D$. In fact, the group interarrivals are distributed according to an entirely different type of PMF, called a geometric, which assigns some positive probability to any interarrival distance that is a multiple of D.

The decomposition assumption is valid, however, for one particular distribution: the standard negative exponential (NE). Thus, since the DNE is very close to being an NE, especially when traffic is sparse (i.e. M is "small" compared to the expected separation), the decomposition of DNE arrivals may not be a bad assumption. In order to quantify the error induced when we decompose DNE arrivals, we need to be able to calculate the actual interarrival distribution for a subgroup within a stream when the overall stream interarrival distance is distributed according to a DNE. This calculation can be made as follows. Let $f^{(k)}(x)$ be the PDF for the k^{th} order interarrival distance in the DNE stream; that is, $f^{(k)}(d)$ is the probability that the distance between any arrival and the k^{th} arrival after it will be between d and d + Δd (for small Δd). Similarly, we define $g^{(k)}(x)$, the PDF for the k^{th} order interarrival distance in the NE stream, where the NE stream has the same expected interarrival distance as the DNE stream. If M is again the delay in the DNE stream, it follows that

$$f^{(k)}(d) = \begin{cases} g^{(k)}(d - kM) & \textit{for } d \geq kM \\ 0 & \textit{otherwise} \end{cases} \qquad (4-13)$$

This result can be established by remembering that the DNE interarrival distance (call it X) can be generated from a NE random variable Y. If E[X] = S + M, and E[Y] = S, then we can generate X by setting

$$X = Y + M. \qquad (4\text{-}14)$$

Thus

$$f^{(1)}(d) = g^{(1)}(d - M) \qquad \text{for} \quad d > M,$$

and the extension to k^{th} order arrivals follows immediately.

We can now compute for subgroup G in the arrival stream

$$
\begin{aligned}
f^{(k)}(d) \;=\;& g^{(1)}(d - M) \cdot p\,(\text{the first arrival is in } G) \\
&+ g^{(2)}(d - 2M) \cdot p\,(\text{the second arrival is in } G \text{ and the first is not}) \\
&+ g^{(3)}(d - 3M) \cdot p\,(\text{the third arrival is in } G \text{ and the first two are not}) \\
&\circ \\
&\circ \\
&\circ \\
&+ g^{(n)}(d - nM) \cdot p\,(\text{the } n^{th} \text{ arrival is in } G \text{ and the first } n - 1 \text{ are not})
\end{aligned}
\tag{4-15}
$$

In this expression, n is the largest integer such that $nM < d$. We assumed that membership in G was assigned randomly with some probability p. Then we have

$$p\,(\text{the } t^{th} \text{ arrival is in } G \text{ and the first } t - 1 \text{ are not}) = p \cdot (1 - p)^{t-1} . \tag{4-16}$$

Theorem 12

Let X be a random variable distributed according to a delayed negative exponential distribution with delay M and mean $M + S$. Let Y be a random variable distributed according to a standard (non-delayed) negative exponential distribution with mean S. Consider a stream of arrivals with interarrival distance X; let each arrival be assigned to group G with probability p. Then $f_G(d)$, the PDF for the interarrival distance between successive member of G is

$$f_G(d) = \sum_{i=1}^{n} p(1-p)^{i-1} \frac{(\frac{1}{S})^i (d - iM)^{i-1} exp\left(- \frac{d - iM}{S} \right)}{(i-1)!} , \tag{4-17}$$

where n is the largest integer such that $nM < d$.

Proof:

This theorem follows from (4-13), (4-15), (4-16), and the fact that the n^{th} order interarrival distance in a stream of NE arrivals with mean separation S is distributed according to PDF

$$h^n(d) = \frac{\left(\frac{1}{S}\right)^n (d)^{n-1} \exp\left(-\frac{d}{S}\right)}{(n-1)!}, \tag{4-18}$$

the familiar n^{th} order Erlangian distribution.

■

The results of Theorem 12 are coded in Program 6 in Appendix C. Program 6 computes the value of the actual PDF for the interarrival distance between two successive members of a randomly assigned subgroup within a DNE arrival stream. The program then compares this PDF to the PDF calculated assuming the arrival distance between subgroup members is DNE (ie. assuming the original DNE stream can be decomposed into DNE subgroup streams). The results of several runs of Program 6 are given in Tables 4-6. In Tables 4-6 the "% ERR" column gives the percentage difference between the approximate (ie. decomposed DNE) and actual (Theorem 12) PDFs at the indicated interarrival distance. The "% TOT ERR" column is computed by summing the values of the actual PDF at .1 NM increments, summing the difference between the actual and approximate PDF observed at each .1 NM increment, and then calculating the ratio of the two sums. This ratio, listed in the % TOT ERR column , is a measure of the difference in the cumulative density functions CDFs) derived by integrating over the two PDFs. This difference in CDFs is the critical value, because our model integrates over the approximate PDF when computing the probability that crossing or overtaking aircraft will or will not be close enough to generate an intervention.

From the data in Tables 4-6 we make the following observations:

1. The error in the approximate CDF (vs. the actual CDF computed from Theorem 12) is a function of both traffic density and the probability that an arrival is in the subgroup of interest.

2. For light traffic, a high probability subgroup will have less than 1% error in its approximate CDF.

3. As traffic density increases and subgroup probability decreases (that is, as arrivals become less likely to belong to that particular subgroup), the approximate CDF error will be greater. At unrealistically high densities (10 NM expected separatin) and low subgroup probabilities (.05) , the CDF error may be as high as 40%.

4. For traffic densities in the range found in the en route structure, the CDF error is in the 1-10% range.

Thus we see that the decomposition assumption can generate errors in approximate CDF, and thus in probability of conflict, on the order of 10% or less . This error will result in an overestimate of the intervention rate in the same 0-10% range.

The reader should note that the approximate PDF tends to overestimate the liklihood of shorter interarrival distances, and underestimates the probability of linger ones. This is consistent with the intuitive expectation that the decomposed PDF will not take account of the imbedded null zones which follow the arrival of aircraft in other subgroups.

TABLES 4-6

EXPECTED DISTANCE BETWEEN ARRIVALS ? 60
P = ? .5
RUN LIMIT IN 5 MILE INCREMENTS ? 30

DISTANCE (NM)	ACTUAL PDF	APPROX PDF	% ERR	% TOT ERR
0	0	0		
5	8.333334E-03	8.333334E-03	0	0
10	7.667037E-03	7.993246E-03	-4.254686E-02	-2.082145E-02
15	7.373475E-03	7.667037E-03	-3.981328E-02	-3.049568E-02
20	7.084497E-03	7.354141E-03	-.0380612	-3.315659E-02
25	6.806931E-03	7.054015E-03	-3.629889E-02	-3.408978E-02
30	6.540239E-03	6.766137E-03	-.0345397	-3.433049E-02
35	6.283997E-03	6.490007E-03	-3.278335E-02	-.0342288
40	6.037793E-03	6.225147E-03	-3.103009E-02	-3.393595E-02
45	5.801236E-03	5.971094E-03	-2.927977E-02	-3.352789E-02
50	5.573946E-03	5.727411E-03	-.0275326	-3.304691E-02
55	5.355562E-03	5.493672E-03	-2.578824E-02	-3.251853E-02
60	5.145734E-03	5.269473E-03	-2.404687E-02	-3.195887E-02
65	4.944127E-03	5.054423E-03	-2.230841E-02	-3.137876E-02
70	4.750419E-03	4.848149E-03	-2.057297E-02	-3.078565E-02
75	.0045643	4.650293E-03	-1.884039E-02	-3.018482E-02
80	4.385473E-03	4.460512E-03	-1.711081E-02	-.0295802
85	4.213653E-03	4.278477E-03	-1.538422E-02	-2.897463E-02
90	4.048564E-03	4.103869E-03	-1.366045E-02	-2.837034E-02
95	3.889944E-03	3.936388E-03	-1.193954E-02	-2.776896E-02
100	3.737538E-03	3.775742E-03	-1.022177E-02	-2.717186E-02
105	3.591103E-03	3.621652E-03	-8.506796E-03	-2.658002E-02
110	3.450405E-03	3.47385E-03	-6.79482E-03	-2.599426E-02
115	3.31522E-03	3.332081E-03	-5.085695E-03	-2.541522E-02
120	3.185332E-03	3.196097E-03	-3.379384E-03	-2.484344E-02
125	3.060532E-03	3.065662E-03	-1.676165E-03	-2.427932E-02
130	2.940623E-03	2.940551E-03	2.454497E-05	-2.372318E-02
135	2.82541E-03	2.820545E-03	1.721873E-03	-2.317527E-02
140	2.714713E-03	2.705438E-03	3.416669E-03	-2.263582E-02
145	2.608352E-03	2.595027E-03	5.10864E-03	-2.210495E-02
150	2.506158E-03	2.489122E-03	6.79728E-03	-2.158282E-02

EXPECTED DISTANCE BETWEEN ARRIVALS ? 60
P = ? .95
RUN LIMIT IN 5 MILE INCREMENTS ? 30

DISTANCE (NM)	ACTUAL PDF	APPROX PDF	% ERR	% TOT ERR
0	0	0		
5	1.583333E-02	1.583333E-02	0	0
10	1.456737E-02	1.462819E-02	-4.175406E-03	-2.056109E-03
15	1.346333E-02	1.351478E-02	-3.822045E-03	-2.976673E-03
20	1.244283E-02	1.248612E-02	-3.479092E-03	-3.181783E-03
25	1.149968E-02	1.153575E-02	-3.136295E-03	-3.209201E-03
30	1.062802E-02	1.065772E-02	-2.793699E-03	-3.167746E-03
35	9.822439E-03	9.846513E-03	-2.450894E-03	-3.093938E-03
40	9.077914E-03	9.097056E-03	-2.108576E-03	-3.003482E-03
45	8.389824E-03	8.404641E-03	-1.766109E-03	-2.904222E-03
50	7.75389E-03	7.76493E-03	-1.423728E-03	-2.800512E-03
55	7.166158E-03	7.173909E-03	-1.081602E-03	-2.694941E-03
60	6.622977E-03	6.627874E-03	-7.3945E-04	-2.589139E-03
65	6.120966E-03	.0061234	-3.976516E-04	-2.484166E-03
70	5.657007E-03	5.657323E-03	-5.581014E-05	-2.380749E-03
75	5.228215E-03	5.226721E-03	2.857269E-04	-2.279377E-03
80	4.831926E-03	4.828894E-03	6.274766E-04	-2.180387E-03
85	4.465674E-03	4.461347E-03	9.689298E-04	-2.084015E-03
90	4.127184E-03	4.121775E-03	1.310383E-03	-1.99042E-03
95	3.81435E-03	3.808051E-03	1.651457E-03	-1.899709E-03
100	3.525229E-03	3.518204E-03	1.992637E-03	-1.811947E-03
105	3.258022E-03	3.25042E-03	2.333579E-03	-1.727166E-03
110	3.01107E-03	3.003017E-03	2.674591E-03	-1.645377E-03
115	2.782836E-03	2.774445E-03	3.01518E-03	-1.566567E-03
120	2.571902E-03	2.563271E-03	3.355986E-03	-1.490714E-03
125	2.376956E-03	2.36817E-03	3.696268E-03	-1.417782E-03
130	2.196787E-03	2.187919E-03	4.03683E-03	-1.347726E-03
135	2.030274E-03	2.021388E-03	4.37697E-03	-1.280493E-03
140	1.876383E-03	1.867531E-03	4.717519E-03	-1.216022E-03
145	1.734156E-03	1.725386E-03	5.057167E-03	-1.154252E-03
150	1.60271E-03	1.59406E-03	5.397114E-03	-1.095115E-03

EXPECTED DISTANCE BETWEEN ARRIVALS ? 60
P = ? .05
RUN LIMIT IN 5 MILE INCREMENTS ? 30

DISTANCE (NM)	ACTUAL PDF	APPROX PDF	% ERR	% TOT ERR
0	0	0		
5	8.333334E-04	8.333334E-04	0	0
10	7.667037E-04	8.298683E-04	-8.238466E-02	-4.006495E-02
15	7.660989E-04	8.264177E-04	-7.873508E-02	-5.937381E-02
20	7.630918E-04	8.229816E-04	-7.848301E-02	-6.564643E-02
25	7.601526E-04	8.195596E-04	-7.815146E-02	-6.874985E-02
30	7.572237E-04	8.161519E-04	-7.782141E-02	-7.056123E-02
35	7.543061E-04	8.127584E-04	-.0774915	-7.172055E-02
40	7.513997E-04	8.093789E-04	-.0771616	-7.250511E-02
45	7.485046E-04	8.060135E-04	-7.683178E-02	-7.305455E-02
50	7.456205E-04	8.026621E-04	-7.650218E-02	-7.344688E-02
55	7.427476E-04	7.993246E-04	-7.617246E-02	-7.372895E-02
60	7.398859E-04	7.96001E-04	-.075843	-7.393075E-02
65	7.370352E-04	7.926913E-04	-7.551345E-02	-7.407231E-02
70	7.341953E-04	7.893952E-04	-.0751843	-7.416758E-02
75	7.313664E-04	7.861129E-04	-7.485504E-02	-7.422643E-02
80	7.285485E-04	7.828443E-04	-7.452602E-02	-7.425615E-02
85	7.257413E-04	7.795892E-04	-7.419711E-02	-.0742623
90	7.229451E-04	7.763477E-04	-7.386816E-02	-7.424908E-02
95	7.201596E-04	7.731196E-04	-7.353935E-02	-7.421969E-02
100	7.173847E-04	7.69905E-04	-.0732108	-7.417672E-02
105	7.146207E-04	7.667037E-04	-7.288201E-02	-7.412223E-02
110	7.118671E-04	7.635158E-04	-7.255378E-02	-7.405791E-02
115	7.091244E-04	7.603411E-04	-7.222525E-02	-.0739852
120	7.06392E-04	7.571796E-04	-7.189716E-02	-7.390507E-02
125	7.036703E-04	7.540313E-04	-7.156891E-02	-7.381859E-02
130	7.009591E-04	7.50896E-04	-.0712407	-7.372651E-02
135	6.982584E-04	7.477737E-04	-7.091254E-02	-.0736295
140	6.955679E-04	7.446645E-04	-.0705849	-7.352815E-02
145	6.928878E-04	7.415681E-04	-.0702571	-7.342296E-02
150	6.902183E-04	7.384848E-04	-6.992932E-02	-7.331431E-02

EXPECTED DISTANCE BETWEEN ARRIVALS ? 20
P = ? .5
RUN LIMIT IN 5 MILE INCREMENTS ? 30

DISTANCE (NM)	ACTUAL PDF	APPROX PDF	% ERR	% TOT ERR
0	0	0		
5	.025	.025	0	0
10	1.947002E-02	2.206242E-02	-.1331485	-6.233585E-02
15	1.759702E-02	1.947002E-02	-.1064385	-8.727141E-02
20	1.575209E-02	1.718223E-02	-9.079066E-02	-9.056611E-02
25	1.410561E-02	1.516327E-02	-7.498113E-02	-8.898349E-02
30	1.263113E-02	1.338154E-02	-5.940915E-02	-8.559776E-02
35	1.131078E-02	1.180917E-02	-4.406294E-02	-8.146296E-02
40	1.012845E-02	1.042155E-02	-2.893879E-02	-7.703185E-02
45	9.069706E-03	9.196986E-03	-1.403365E-02	-7.252781E-02
50	8.121635E-03	8.116313E-03	6.553494E-04	-6.807104E-02
55	7.272668E-03	7.162621E-03	1.513164E-02	-6.372973E-02
60	6.512444E-03	6.32099E-03	2.939823E-02	-5.954327E-02
65	5.831689E-03	5.578255E-03	4.345813E-02	-5.553438E-02
70	5.222093E-03	4.922792E-03	5.731438E-02	-5.171533E-02
75	4.676221E-03	4.344349E-03	7.097007E-02	-4.809175E-02
80	4.187408E-03	3.833874E-03	8.442768E-02	-4.466475E-02
85	3.749691E-03	3.383382E-03	9.769049E-02	-4.143256E-02
90	3.35773E-03	2.985824E-03	.1107612	-3.839126E-02
95	3.006742E-03	2.634981E-03	.1236425	-3.553555E-02
100	2.692442E-03	2.325362E-03	.1363372	-3.285912E-02
105	2.410998E-03	2.052125E-03	.1488481	-3.035503E-02
110	2.158972E-03	1.810994E-03	.1611776	-2.801587E-02
115	1.933292E-03	1.598197E-03	.1733286	-2.583402E-02
120	1.731202E-03	1.410404E-03	.1853036	-2.380173E-02
125	1.550237E-03	1.244677E-03	.1971052	-2.191123E-02
130	1.388188E-03	1.098424E-03	.2087358	-2.015481E-02
135	1.243079E-03	9.693552E-04	.220198	-1.852492E-02
140	1.113138E-03	8.55453E-04	.2314941	-1.701416E-02
145	9.967796E-04	7.549346E-04	.2426264	-1.561536E-02
150	8.925847E-04	6.662276E-04	.2535974	-1.432157E-02

EXPECTED DISTANCE BETWEEN ARRIVALS ? 20
P = ? .95
RUN LIMIT IN 5 MILE INCREMENTS ? 30

DISTANCE_(NM)	ACTUAL_PDF	APPROX_PDF	%_ERR	%_TOT_ERR
0	0	0		
5	.0475	.0475	0	0
10	3.699304E-02	3.745835E-02	-1.257844E-02	-6.00416E-03
15	2.927262E-02	2.953954E-02	-9.118445E-03	-8.112256E-03
20	2.316056E-02	2.329479E-02	-5.795695E-03	-7.952591E-03
25	1.832469E-02	.0183702	-2.483433E-03	-7.316073E-03
30	1.449854E-02	1.448668E-02	8.181688E-04	-6.557848E-03
35	1.147129E-02	1.142415E-02	4.108727E-03	-5.793097E-03
40	9.076109E-03	9.009051E-03	7.388411E-03	-5.067243E-03
45	7.181041E-03	7.104509E-03	1.065749E-02	-4.399132E-03
50	5.681659E-03	5.602594E-03	1.391585E-02	-3.795624E-03
55	4.495343E-03	4.418188E-03	.0171631	-3.257596E-03
60	3.556727E-03	3.48417E-03	2.039981E-02	-2.782747E-03
65	2.814092E-03	2.747606E-03	.023626	-2.367084E-03
70	2.226516E-03	2.166753E-03	2.684158E-02	-2.005757E-03
75	1.761625E-03	1.708695E-03	.0300463	-1.693571E-03
80	1.393803E-03	1.347472E-03	.0332407	-1.425302E-03
85	1.10278E-03	1.062612E-03	3.642468E-02	-1.195895E-03
90	8.725223E-04	8.379723E-04	3.959788E-02	-1.00059E-03
95	6.90342E-04	6.608222E-04	.0427611	-8.349897E-04
100	5.462004E-04	5.211226E-04	4.591315E-02	-6.951014E-04
105	4.321551E-04	4.109556E-04	4.905529E-02	-5.773363E-04
110	3.419222E-04	3.240783E-04	5.218704E-02	-4.785092E-04
115	2.705298E-04	2.555672E-04	5.530852E-02	-3.958164E-04
120	2.140439E-04	2.015394E-04	5.842002E-02	-3.268104E-04
125	1.693521E-04	1.589334E-04	6.152056E-02	-2.693703E-04
130	1.339918E-04	1.253343E-04	6.461165E-02	-2.216681E-04
135	1.060146E-04	9.883829E-05	6.769194E-02	-1.821385E-04
140	8.387905E-05	7.794356E-05	7.076236E-02	-1.494467E-04
145	6.63653E-05	6.146607E-05	7.382216E-02	-1.224603E-04
150	5.250842E-05	4.847195E-05	7.687272E-02	-1.002222E-04

EXPECTED DISTANCE BETWEEN ARRIVALS ? 20
P = ? .05
RUN LIMIT IN 5 MILE INCREMENTS ? 30

DISTANCE (NM)	ACTUAL PDF	APPROX PDF	% ERR	% TOT ERR
0	0	0		
5	.0025	.0025	0	0
10	1.947002E-03	2.468945E-03	-.2680751	-.1230519
15	1.97874E-03	2.438275E-03	-.2322364	-.1787141
20	1.956083E-03	2.407986E-03	-.2310244	-.1957057
25	1.936509E-03	2.378074E-03	-.2280207	-.2037739
30	1.917068E-03	2.348533E-03	-.2250649	-.2081246
35	1.89782E-03	2.319359E-03	-.2221176	-.210583
40	1.878765E-03	2.290547E-03	-.2191774	-.2119497
45	1.859901E-03	2.262094E-03	-.2162442	-.2126306
50	1.841227E-03	2.233993E-03	-.2133178	-.2128539
55	1.82274E-03	2.206242E-03	-.2103987	-.2127572
60	1.804439E-03	2.178836E-03	-.2074866	-.212429
65	1.786321E-03	2.15177E-03	-.2045818	-.2119282
70	1.768386E-03	2.12504E-03	-.2016835	-.2112962
75	1.750631E-03	2.098643E-03	-.1987924	-.2105618
80	1.733053E-03	2.072573E-03	-.1959084	-.2097471
85	1.715653E-03	2.046827E-03	-.1930309	-.2088682
90	1.698427E-03	2.021401E-03	-.1901607	-.2079376
95	1.681374E-03	1.99629E-03	-.1872972	-.2069649
100	1.664492E-03	1.971492E-03	-.1844409	-.2059581
105	1.64778E-03	1.947002E-03	-.1815913	-.2049231
110	1.631235E-03	1.922816E-03	-.1787486	-.2038648
115	1.614857E-03	1.89893E-03	-.1759126	-.2027875
120	1.598643E-03	1.875342E-03	-.1730834	-.2016944
125	1.582592E-03	1.852045E-03	-.1702609	-.2005884
130	1.566702E-03	1.829039E-03	-.1674455	-.1994719
135	1.550971E-03	1.806318E-03	-.1646369	-.1983466
140	1.535399E-03	1.78388E-03	-.1618349	-.1972144
145	1.519983E-03	1.76172E-03	-.1590396	-.1960766
150	1.504721E-03	1.739836E-03	-.1562511	-.1949347

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EXPECTED DISTANCE BETWEEN ARRIVALS ? 10
P = ? .5
RUN LIMIT IN 5 MILE INCREMENTS ? 30
```

DISTANCE (NM)	ACTUAL PDF	APPROX PDF	% ERR	% TOT ERR
0	0	0		
5	.05	.05	0	0
10	3.032654E-02	3.894004E-02	-.2840254	-.1240035
15	2.597561E-02	3.032654E-02	-.1675006	-.1608453
20	.0213012	2.361833E-02	-.1087794	-.1553848
25	1.751237E-02	1.839397E-02	-5.034196E-02	-.1422379
30	1.439693E-02	1.432524E-02	4.979112E-03	-.1273257
35	1.183564E-02	1.115651E-02	5.737992E-02	-.1125819
40	9.730021E-03	8.688698E-03	.1070217	-9.874383E-02
45	7.999005E-03	6.766764E-03	.1540493	-8.609094E-02
50	6.575944E-03	5.269962E-03	.1985999	-7.470188E-02
55	5.406053E-03	4.10425E-03	.2408047	-6.455913E-02
60	4.44429E-03	3.196393E-03	.2807865	-5.559778E-02
65	3.65363E-03	2.489354E-03	.3186629	-4.773008E-02
70	3.003632E-03	1.93871E-03	.3545447	-4.085893E-02
75	2.469272E-03	1.509869E-03	.3885367	-3.488537E-02
80	2.029977E-03	1.175888E-03	.4207384	-2.971308E-02
85	1.668834E-03	9.15782E-04	.4512444	-2.525077E-02
90	1.371941E-03	7.132117E-04	.4801439	-2.141369E-02
95	1.127866E-03	5.554499E-04	.5075213	-1.812421E-02
100	9.272132E-04	4.325848E-04	.533457	-1.531204E-02
105	7.622576E-04	3.368973E-04	.558027	-1.291412E-02
110	6.266485E-04	2.62376E-04	.5813028	-1.087432E-02
115	5.151647E-04	2.043386E-04	.6033528	-9.143026E-03
120	4.235145E-04	1.591391E-04	.6242416	-7.676617E-03
125	3.481692E-04	1.239376E-04	.6440304	-6.436958E-03
130	2.862282E-04	9.652272E-05	.6627771	-5.390872E-03
135	2.353069E-04	7.517196E-05	.6805365	-4.509604E-03
140	1.934447E-04	5.854398E-05	.6973606	-3.768352E-03
145	1.5903E-04	4.559409E-05	.7132988	-3.145774E-03
150	1.307378E-04	3.550873E-05	.7283974	-2.623579E-03

EXPECTED DISTANCE BETWEEN ARRIVALS ? 10
P = ? .95
RUN LIMIT IN 5 MILE INCREMENTS ? 30

DISTANCE (NM)	ACTUAL PDF	APPROX PDF	% ERR	% TOT ERR
0	0	0		
5	.095	.095	0	0
10	5.762042E-02	5.907908E-02	-2.531511E-02	-1.151387E-02
15	3.638906E-02	.0367404	-9.655192E-03	-1.386584E-02
20	.0229628	2.284831E-02	4.986024E-03	-1.172982E-02
25	1.449049E-02	1.420902E-02	1.942459E-02	-9.187048E-03
30	9.144107E-03	8.836376E-03	3.365341E-02	-6.932203E-03
35	5.770316E-03	5.495212E-03	4.767571E-02	-5.104721E-03
40	3.641311E-03	3.41739E-03	6.149454E-02	-3.690554E-03
45	2.29782E-03	2.125224E-03	7.511313E-02	-2.628931E-03
50	1.450021E-03	1.321644E-03	8.853412E-02	-1.849719E-03
55	9.150238E-04	8.219112E-04	.1017598	-1.287906E-03
60	5.774184E-04	5.111344E-04	.1147939	-8.887174E-04
65	3.643753E-04	3.178669E-04	.1276389	-6.085212E-04
70	2.299361E-04	1.976766E-04	.1402976	-4.138702E-04
75	1.450993E-04	1.229321E-04	.1527721	-2.798343E-04
80	9.156374E-05	7.644966E-05	.1650661	-1.882346E-04
85	5.778056E-05	4.754289E-05	.177182	-1.26045E-04
90	3.646195E-05	2.956622E-05	.1891213	-8.406296E-05
95	2.300903E-05	1.838678E-05	.2008884	-5.586368E-05
100	1.451966E-05	1.143447E-05	.2124832	-3.700542E-05
105	9.162511E-06	7.110925E-06	.223911	-2.444298E-05
110	5.78193E-06	4.422179E-06	.2351725	-1.610347E-05
115	3.64864E-06	2.750088E-06	.2462706	-1.058448E-05
120	2.302446E-06	1.710238E-06	.2572082	-6.94225E-06
125	1.45294E-06	1.063572E-06	.2679861	-4.544617E-06
130	9.168658E-07	6.614189E-07	.2786089	-2.969895E-06
135	5.785808E-07	4.113266E-07	.2890766	-1.937788E-06
140	3.651086E-07	2.557979E-07	.2993924	-1.262614E-06
145	2.303988E-07	1.590769E-07	.3095583	-8.217026E-07
150	1.453913E-07	9.892759E-08	.3195771	-5.342349E-07

```
EXPECTED DISTANCE BETWEEN ARRIVALS ? 10
P = ? .05
RUN LIMIT IN 5 MILE INCREMENTS ? 30
```

DISTANCE (NM)	ACTUAL PDF	APPROX PDF	% ERR	% TOT ERR
0	0	0		
5	.005	.005	0	0
10	3.032653E-03	4.87655E-03	-.6080144	-.2544749
15	3.279908E-03	4.756147E-03	-.4500857	-.3568044
20	3.205199E-03	4.638718E-03	-.4472479	-.3866969
25	3.150676E-03	4.524187E-03	-.4359418	-.3993986
30	3.098119E-03	4.412485E-03	-.4242462	-.4050811
35	3.046181E-03	4.30354E-03	-.4127659	-.4071411
40	2.995123E-03	4.197285E-03	-.4013735	-.4071177
45	2.944922E-03	4.093654E-03	-.3900722	-.4057966
50	2.895562E-03	3.992581E-03	-.378862	-.4036215
55	2.84703E-03	3.894004E-03	-.3677423	-.4008613
60	2.799312E-03	3.797861E-03	-.3567123	-.3976885
65	2.752393E-03	3.704091E-03	-.3457713	-.3942195
70	2.706261E-03	3.612637E-03	-.3349184	-.3905343
75	2.660901E-03	3.52344E-03	-.3241532	-.3866907
80	2.616302E-03	3.436447E-03	-.3134745	-.3827312
85	2.572451E-03	.0033516	-.3028821	-.3786872
90	2.529334E-03	3.268849E-03	-.2923752	-.3745832
95	2.486941E-03	3.188141E-03	-.281953	-.3704379
100	2.445257E-03	3.109425E-03	-.2716149	-.366266
105	2.404273E-03	3.032653E-03	-.26136	-.3620794
110	2.363975E-03	2.957777E-03	-.2511881	-.3578872
115	2.324352E-03	2.884749E-03	-.241098	-.353697
120	2.285395E-03	2.813524E-03	-.231089	-.3495152
125	2.24709E-03	2.744058E-03	-.2211609	-.3453472
130	2.209426E-03	2.676307E-03	-.2113132	-.3411969
135	2.172395E-03	2.610229E-03	-.2015446	-.3370681
140	2.135983E-03	2.545782E-03	-.1918548	-.3329634
145	2.100183E-03	2.482927E-03	-.182243	-.3288852
150	2.064982E-03	2.421623E-03	-.172709	-.3248358

Chapter 5
Extensions to the Two-Dimensional Model

5.1 A Three-Dimensional High Altitude Sector

The objective of this chapter is to show some possible extensions of the model developed in Chapter 2. We begin by discussing the possibility of extending the model to three dimensions, in order to represent the total airspace for which a typical en route controller will have responsibility.

Ideally, we would like to be able to compute the intervention rate for the entire three-dimensional sector by summing the intervention rates computed by our two-dimensional model for each of the separate two-dimensional flight levels in the sector. This would be quite feasible if it were not for the fact that traffic frequently passes between flight levels. These altitude changes may involve aircraft transitioning to or from take-off or landing, or perhaps aircraft requesting a new altitude due to adverse weather or for more fuel efficient cruise. In any case, such climbing/descending traffic must be taken into account, since they certainly contribute to the number of controller interventions required to avoid potential conflicts.

First we will consider climbing/descending traffic that is randomly distributed horizontally; that is, traffic equally likely to appear at any point on the controller's (two-dimensional) radar scope. This type of traffic will generate interventions due to crossing conflicts both with other such randomly distributed climbing/descending aircraft as well as with traffic in level flight, confined to one particular altitude.

The technique we will use to model this traffic will be to include it in the level flight traffic already modeled in Chapter 2, by suitably modifying the network geometry and traffic density parameters in each two-dimensional flight level. To do this we need to determine what portion of the total climbing/descending traffic should be associated with each individual flight level. We begin by considering that portion of the climbing/descending traffic not associated with a published airway.

Consider a situation in which on average N aircraft are climbing or descending off-airways in a given three-dimensional sector. Let the sector have horizontal area A, and be comprised of M flight levels. If the aircraft are distributed uniformly in the vertical, then we can associate N/M of the off-airway climbing/descending aircraft with each of the M flight levels. This last assumes that the climbing/descending traffic interact with only one flight level at a time – that is, they are in potential conflict with at most one flight level at any one time. This would be the case if the climbing/descending aircraft were considered to be a hazard only when they were separated by less than 500 feet vertically from level flight aircraft. If 1000 feet vertical separation were required, then the climbing/descending aircraft would be conflicting with two flight levels simultaneously, so we would associate 2N/M of them with each flight level.

Assuming the 1000 feet separation case, we would then simply add 2N/ (MA) to the off-airway level flight traffic density for each flight level, and then compute crossing conflicts for each flight level using the augmented off-airway traffic density and the model from Section 2.3. In other words, we split the climbing/descending traffic up among the several flight levels in a natural way, and then compute intervention rates at each flight level as if the associated climbing/descending aircraft were in level flight.

We use the same approach to model the effect of climbing/descending traffic that is clustered about commonly used but unpublished routing. Consider the single airway segment shown in Figure 5-1. In general, there are two ways that an unpublished climb/descent route could

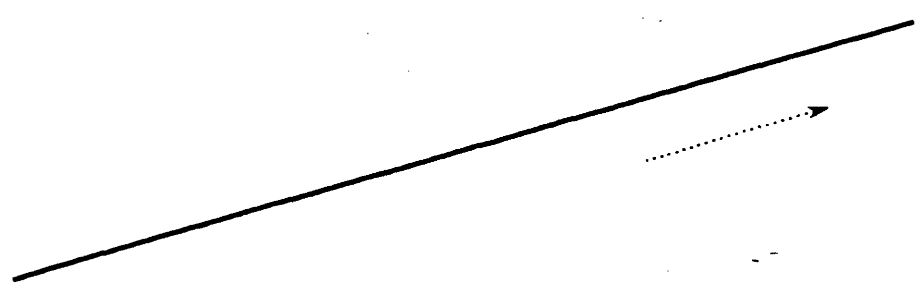


Figure 5-1

A Single Airway Segment

interact with this airway: by crossing it, or by merging with it. In Figure 5-2 we see the first of these types, the crossing route.

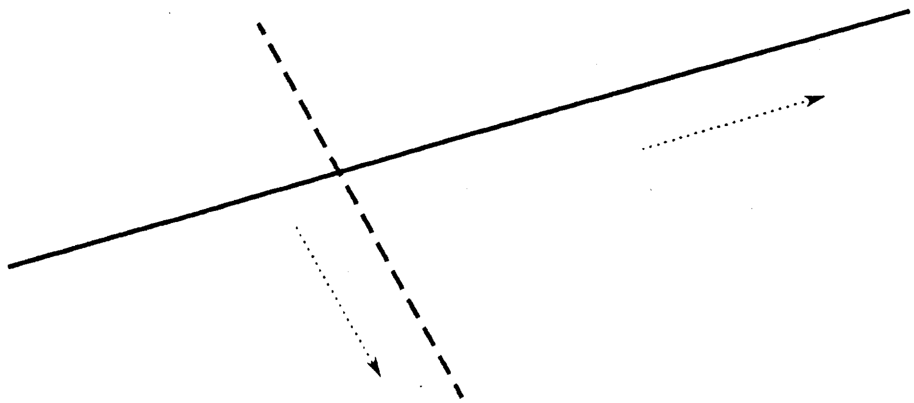


Figure 5-2
A Single Airway Segment With a Crossing Climb/Descent Route

A natural way to model the crossing route is to consider it to be another published airway at the altitude in question, and treat it just like the "horizontal airways" we have discussed in detail in Chapter 2. We can compute the number of interventions due to the new "airway" just as we did for the horizontal routes, with the only question being, just what is the length of the crossing segment? One way to determine the length of the climb/descent segment is to determine over what portion of its total length (which may encompass several flight levels) the traffic on the route will conflict vertically with aircraft maintaining a constant altitude on the published airway. We might, for example, consider a climbing or descending aircraft to be a potential conflict anytime its altitude is within 1000 feet of the altitude flown by the horizontal traffic. Notice that this technique may count more than one intervention involving the climbing / descending aircraft , if it conflicts with one or more other aircraft at more than one altitude. At reasonable traffic densities the error induced thereby is small relative to the more significant sources of error discussed in Chapter 4. We determine the length of the climb/descent route at each flight level to be that portion of its total length over which its traffic will be a potential hazard to horizontal aircraft at that flight level.

There will be some uncertainty involved in determining the exact points along a climb/descent route where traffic will be in potential conflict with aircraft at a particular altitude; thus the exact end points of the segments assigned to each flight level may be difficult to determine. The effect of this uncertaintly may be reduced by the fact that overtaking conflicts on the climb/descent route may be computed independently of the altitude on the route, while crossing conflicts are a function only of the published airways that are crossed by the route. In other words, the overtaking intervention rate on the climb/descent route may be computed by regarding it as a constant altitude airway segment with a particular traffic density and airspeed distribution. The rate at which overtaking conflicts may occur on a climb/descent route are independent of whatever horizontal airways the route may be crossing. On the other hand, the crossing conflict rate generated by superimposing segments of the climb/descent route on successive flight levels is a function only of the segments crossed: the length of the superimposed segment is immaterial.

An example may help clarify this point. Consider the situation in Figure 5-2, where a published airway is being crossed by a flow of descending traffic. The crossing interventions generated by the descending traffic at this flight level will be the same, whether the length of the superimposed segment (the dashed line) is as portrayed in Figure 5-2, or as in Figures 5-3.1 or 5-3.2. Only in a case like the one in Figure 5-4, where the descending traffic is not now considered

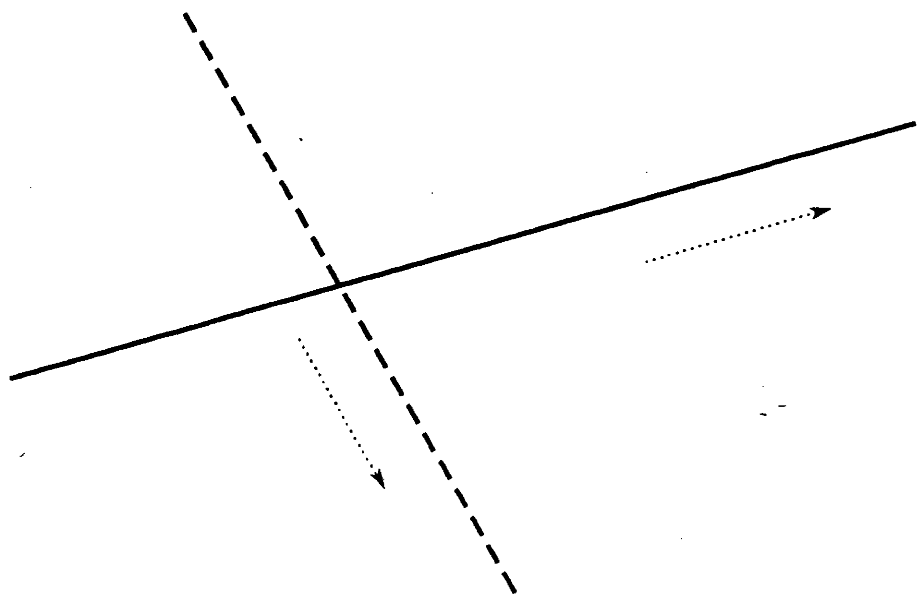


Figure 5-3.1
Variations on a Single Airway Segment With a Crossing Climb/Descent Route

to cross the published segment at all, will the results of the model (and the real-world number of interventions) change. Thus, the effect of the uncertainty involved in determining exactly where a climb/descent route enters and leaves each flight level is diminished by the fact that all

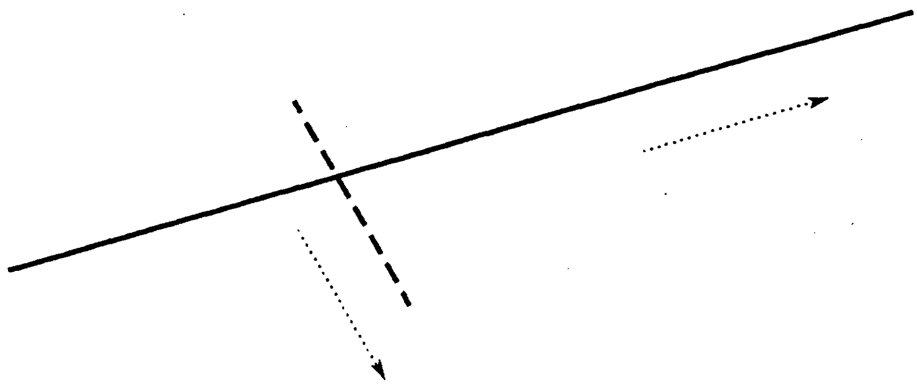


Figure 5-3.2
Variations on a Single Airway Segment With a Crossing Climb/Descent Route

we need to know is which airways the climb/descent routes will cross as it traverses the several flight levels.

The second type of possible interaction between a climb/descent route and published horizontal airways is a bit more complex. This type occurs when traffic is climbing or descending along a published airway, where this airway is published in each of the flight levels penetrated by the climb/descent route. This type of interaction brings up two complications not present in the crossing situation in Figures 5-2 and 5-3:

(1) The end points of the overlap between the climb/descent route and the published horizontal airway may not fall at intersections. Thus we may have to model a traffic source and sink at some point in the middle of an airway segment. This is a new demand, not previously placed upon the model.

(2) Since overtaking conflicts can now occur between climbing/descending and constant altitude traffic, the determination of the end points of the overlap segments becomes more critical.

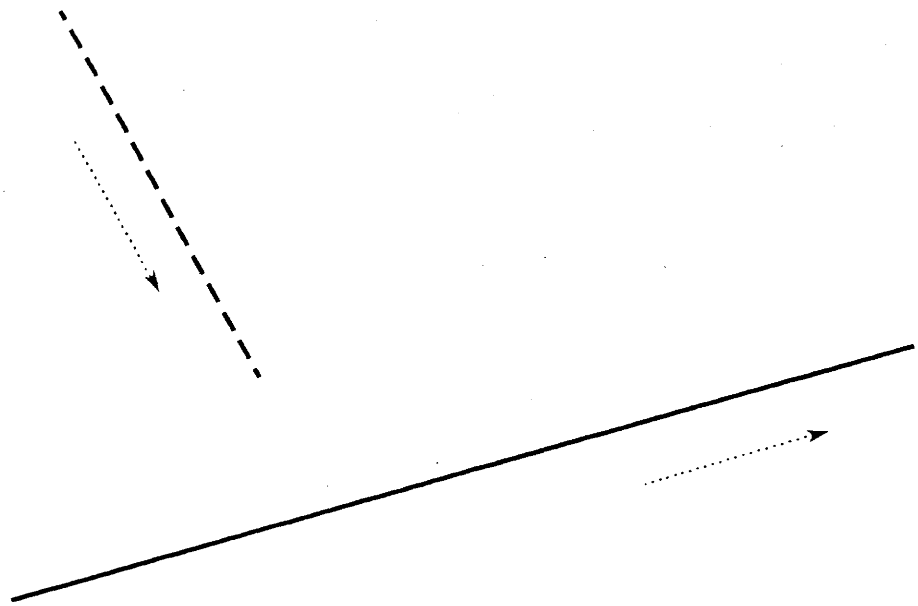


Figure 5-4
Variations on a Single Airway Segment With a Crossing Climb/Descent Route: No Intersection

We will represent this type of interaction pictorially as in Figures 5-5.1 and 5-5.2. We begin with an airway segment (4-5) , and then superimpose descending traffic as portrayed by the "number 7 arrows" which indicate the points where overlap between level and descending traffic begins and ends. We define "overlap" between the level and descending airways to be that portion of their length over which aircraft on the two airways could be in vertical conflict with one-another (ie. when their vertical separtion will be less then 1000 feet). Figure 5-5.3 shows what is happening from a vertical perspective.

The model developed in Chapter 2 can be adapted to include traffic "sources" and "sinks" along published airways by the simple expedient of considering each source or sink as an intersection. For example, the situation pictured in Figure 5-5.2 can be compared to the two-dimensional network shown in Figure 5-6.

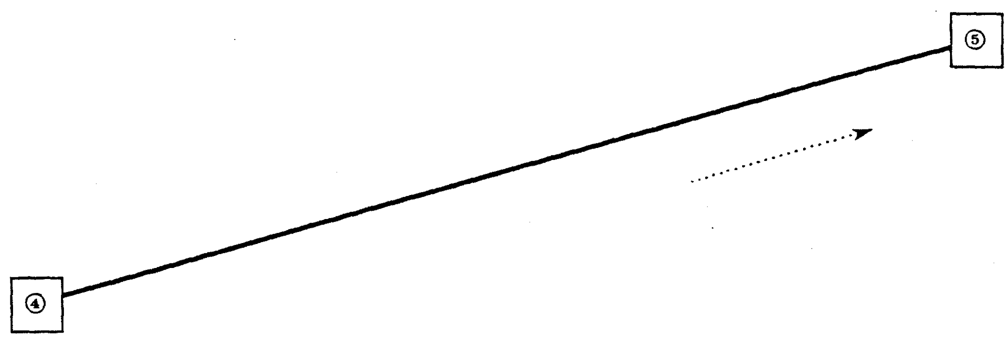


Figure 5-5.1
Airway Segment 4-5
Horizontal View

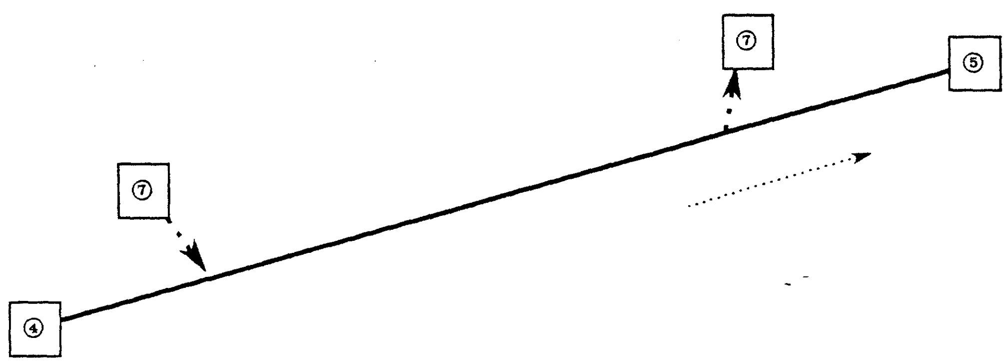


Figure 5-5.2
Airway Segment 4-5 With Descending Traffic
Horizontal View

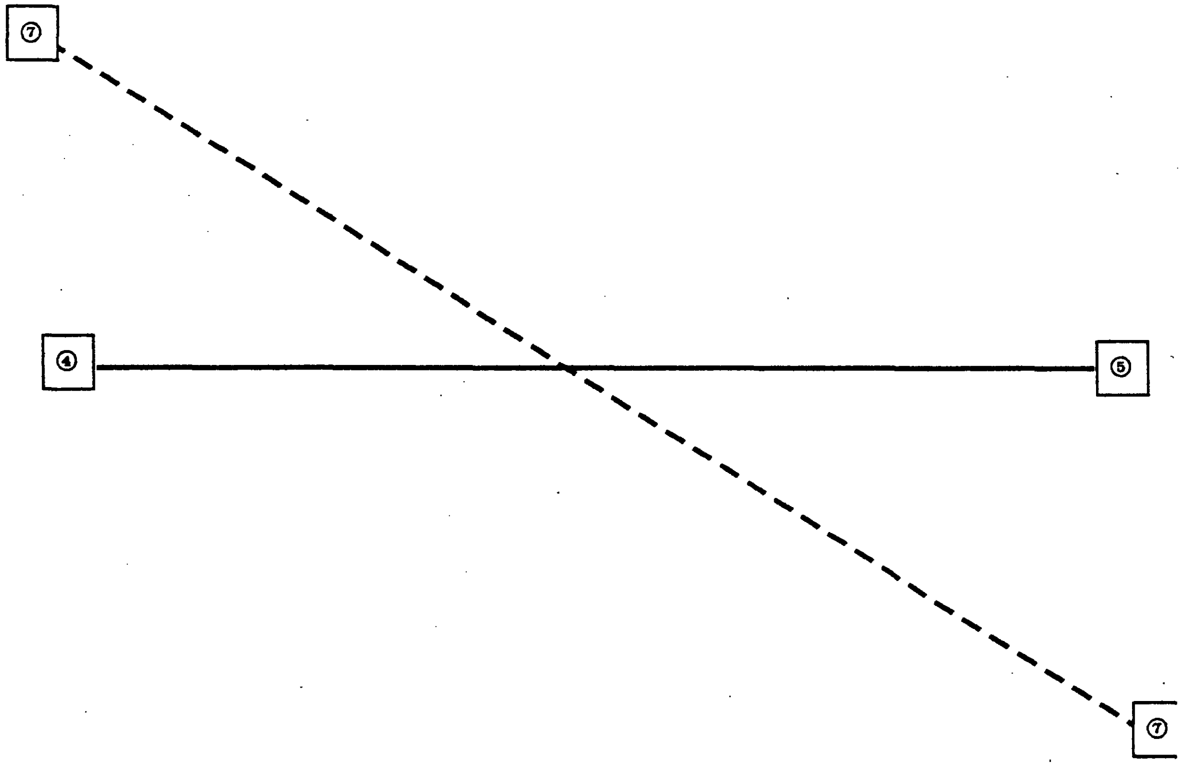


Figure 5-5.3
Airway Segment 4-5 With Descending Traffic
Vertical View

If the flow on airway 4 - 5 (Figure 5-5.2) is the same as the flow on pseudo-airway 4 - 1 - 2 - 5 (Figure 5-6), and if the flow along 7 - 1 - 2 - 7' (Figure 5-6) is the same as the descending flow along 7 - 7 (Figure 5-5.2), then the intervention rate for the two-dimensional network in Figure 5-6 will be nearly the same as the rate for the three-dimensional network in Figure 5-5.2. The rate for Figure 5-6 will be slightly higher than the rate for Figure 5-5.2, because the former will include overtakes along 7 - 1 and 2 - 7', as well as crossing conflicts generated by aircraft on 7 - 1 or 2 - 7'. These cases correspond to horizontal conflicts involving aircraft on the climb/descent path before they conflict vertically with aircraft in level flight along airway 4 - 5.

The overestimate produced when we apply the model to Figure 5-6 can be corrected easily. The count of overtakes along 7 - 1 and 2 - 7' can be deleted from the model simply by removing the corresponding block of code from the program written for the two-dimensional case. As an example, consider Program 2 in Appendix 3 which implements the model for the network in Figure 2-25. If we remove lines 5880-6040 from the program, it will no longer include in its

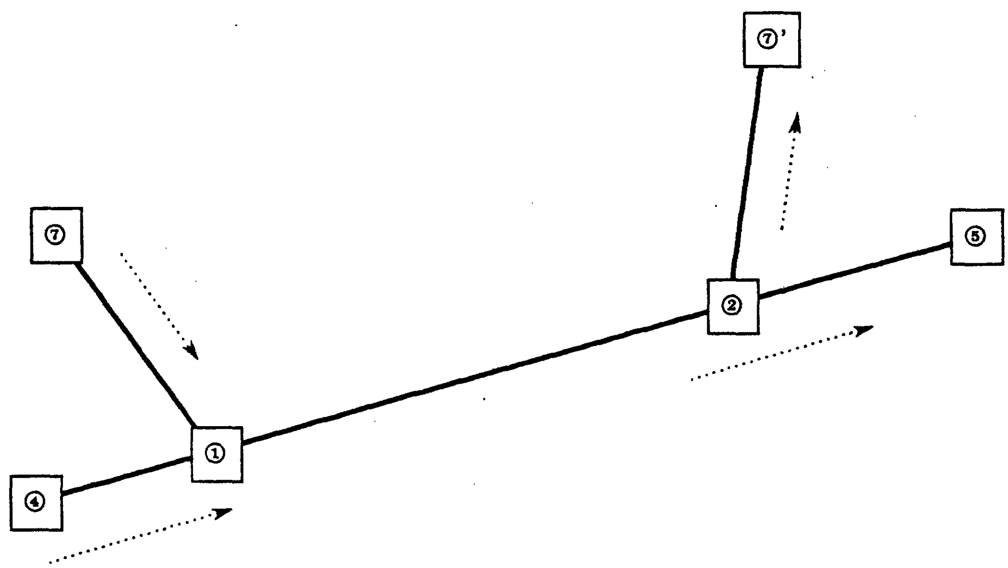


Figure 5-6
Airway Segment 4-5 With Descending Traffic - Horizontal View
(A Two-Dimensional Analog)

computations the overtakes along airway segment 2 - 3 . The program output would then reflect the total sector intervention rate <u>less</u> the overtake rate on 2 - 3.

In a similar fashion we can implement the model in such a way that aircraft on 2 - 7' (Figure 5-6) will not be counted as being involved in a crossing conflict with an aircraft on 4 - 5, and aircraft on 7 - 1 will only be counted if the conflict (i.e. the violation of minimum horizontal separation) occurs while the aircraft on segments 7 - 1 or 2 - 7' is also on 1 - 2 . This is done by disregarding one of the cases in Theorem 3, in much the same way as crossing conflicts were computed between pseudo-airways of Class I or Class II in Theorems 6 and 7 .

The uncertainty surrounding the location of the source and sink in Figure 4-5 is not significant if they are known with reasonable certainty to fall within the limits of a single published airway (as in Figure 5-7). The only error in this case will be due to the uncertain distance between the source and the sink, which would affect the overtake rate between

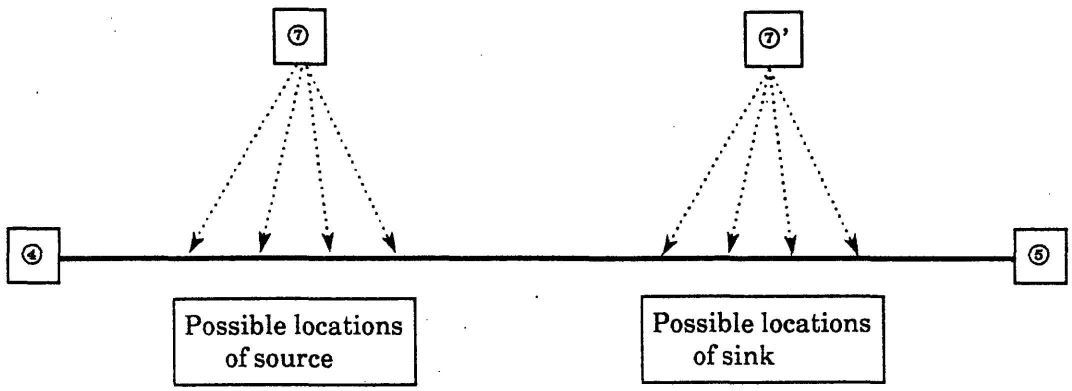


Figure 5-7

Uncertainty Over Source / Sink Location

climbing/descending and level traffic along 1 - 2 . At realistic traffic densities, this factor should be negligible.

In situations where the range of source or sink locations span a published intersection, the geometry of the two-dimensional network analog may change significantly. One may need to look at several two-dimensional analogs to produce a bound on the intervention rate at that altitude. Once again, at realistic traffic densities the difference in predicted intervention rates should fall within the uncertainty resulting from velocity and traffic density variation.

Thus we see that with the use of two-dimensional analogs we can compute the intervention rate due to conflicts involving aircraft climbing or descending along published (or even simply commonly used) tracks. This , in turn, allows us to model the controller intervention rate in a three-dimensional sector by summing across the rates in the (appropriately modified) two-dimensional sectors included therein.

5.2 Three-Dimensional Low Altitude Sectors and Terminal Control Areas

In the previous section we showed that the two-dimensional model developed in Chapter 2 can be extended to describe a three-dimensional en route sector. The techniques described in Section 5.1 can be used to model low alitiude sectors as well, by simply reflecting the increased density of off-airway VFR traffic typically found at lower altitudes.

A natural extention of the three-dimensional low altitude sector model would be to attempt to include altitudes right down to the ground, thereby modeling traffic flow in a Terminal Control Area (TCA). This would be quite feasible if the TCA contained only small, uncontrolled airports where arrivals and departures could be considered to be random events. Unfortunately, TCAs are typically constructed around large commercial airports where neither arrivals nor departures are at all random: they are in fact very carefully controlled. Arriving aircraft are manipulated, often beginning many miles before they actually arrive in the terminal area, to produce a regularly spaced, decidedly non-random flow into the airfield. Similarly, departing aircraft must be released by departure control before they are allowed to take-off, so the flow of recently airborne traffic into the TCA is also highly non-random : controllers intentionally release aircraft so as to enable a smooth, conflict-free transition into the high altitude structure.

Thus it appears that this model, as it stands. is less applicable in the vicinity of large commercial airfields because of the existence of this non-random traffic flow. Extensions of the model to include such situations will be discussed in Chapter 6 as opportunities for further research.

5.3 An Example

In this section we will give an example of a three-dimensional sector and show how the techniques discussed in Section 5.1 can be applied. Assume that each flight level in the sector has the airway network shown in Figure 5-8, and that the sector is 8,000 feet deep. Thus it will

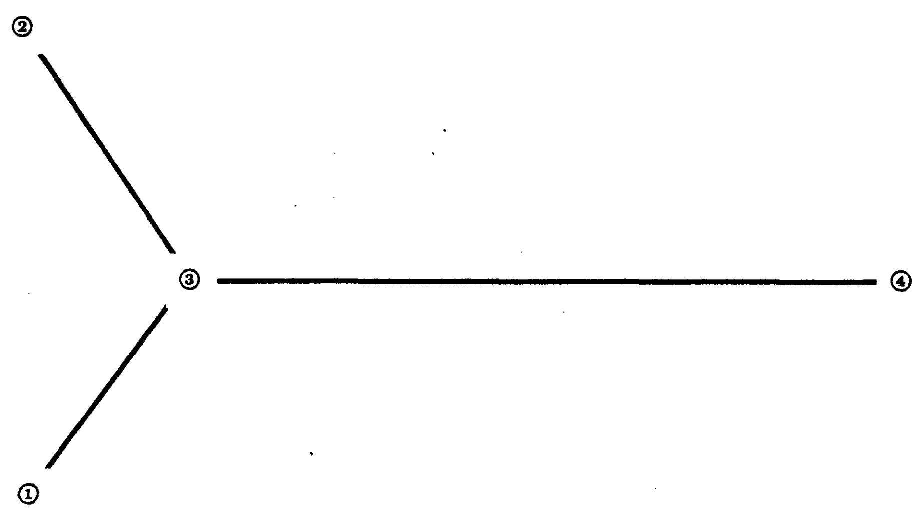


Figure 5-8

have nine distinct "assignable" flight levels, which we will call flight levels 210 through 290. Flight levels (FLs) are named with the first three digits of their corresponding altitude. FL 210, for example, is at approximately 21,000 feet. For each flight level we have corresponding airspeed distribution and traffic density information on the <u>constant altitude traffic</u>. If there were no climbing/descending traffic in this sector, we would now have sufficient information to compute the intervention rate for the entire sector: we would simply compute the rate for each individual (two-dimensional) flight level, and then add the rates for the nine flight levels together.

We assume, however that some additional aircraft in the sector are changing altitude. We begin by noting N, the average number of climbing or descending off-airway aircraft in the sector at any one time. If the horizontal area of the sector is A (we assume the area is the same at all flight levels), then

$$\rho_i = k_i \cdot \frac{N}{A}$$

is the density (in aircraft per NM^2) of off-airway climbing/descending traffic at flight level i. The quantity k_i depends on two factors, the distribution of climbing/descending traffic among the nine flight levels, and the vertical separation from the flight level needed before a climbing/descending aircraft is no longer considered to be of concern. If the distribution of the off-airway climbing/descending traffic is uniform among the nine flight levels, and if traffic is no longer a factor when it is more than 500 feet above or below a given flight level, then $k_i = 1/9$ for all i. If 1000 feet separation is needed for vertical separation, and the traffic is still uniformly distributed among the flight levels, then $k_i = 2/9$ for all i (we neglect boundary problems here - FL 210 and 290 may have slightly different k_i). Once we have the nine ρ_i, we add them to the respective off-airway level flight traffic densities and then recompute R_c for each flight level using the augmented densities according to the formulas derived in Section 2.3. The new R_c's now reflect interventions due to crossing conflicts generated by off-airway climbing/descending traffic in the three-dimensional sector.

Up to this point we have assumed that the off-airway climbing/descending traffic was randomly distributed horizontally throughout the sector. If such were not the case, then we could apply a slightly different technique to include them in the model. Lets assume that a certain percentage of the off-airway climbing/descending traffic tended to follow the ground track indicated by the dashed line in Figure 5-9. Assume further that the stream of climbing/descending traffic following the unpublished routing (the dashed line) tended to be uniformly distributed between FL 240 and FL 290 as they crossed airway 3-4. Then to incorporate the interventions generated at the intersection of the unpublished routing and airway 3-4, we would need to know the flow rate along the unpublished route, the velocity distribution along that route, and the rate of climb or descent being maintained by the aircraft using it. This last is important because it allows us to determine how many flight levels a given climbing/descending aircraft may potentially conflict with as it crosses airway 3-4.

An example may clarify this last point. If an aircraft descending along the unpublished route is descending at a relatively low rate (less than 500 feet per minute), it will travel approximately 12-15 miles horizontally for every thousand feet it will descend vertically. Thus, while it is in close proximity to airway 3-4, it will most likely conflict with (that is, be within 1000 feet vertically of) at most two or three different flight levels. As the vertical velocity of the aircraft

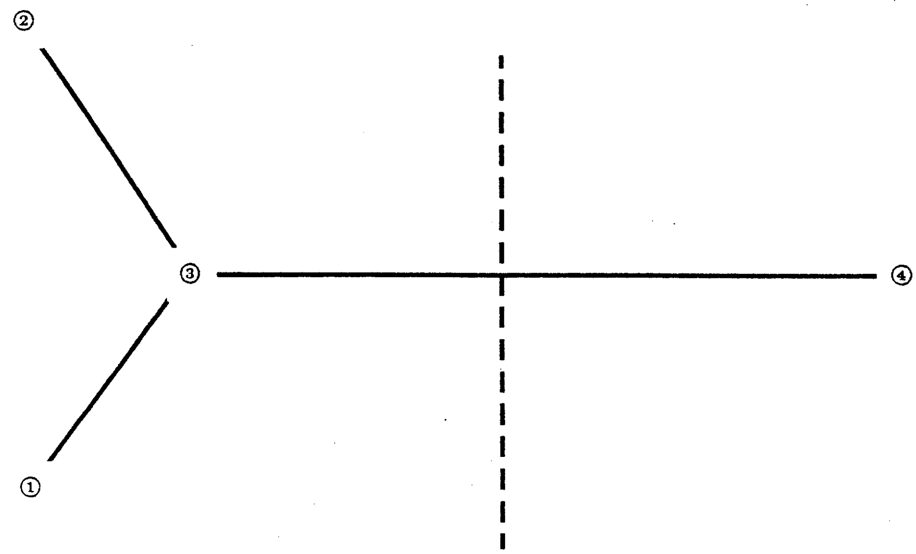


Figure 5-9

increases, it will potentially conflict with more individual flight levels in the sector. While most commercial aircraft are capable of fairly high rates of climb or descent (as much as several thousand feet per minute), lower rates, particularly for passenger aircraft, are more common. For typical high altitude traffic, climb/descent rates resulting in potential conflict with three flight levels is probably most representative. For this example, we'll assume such a rate.

To model this climb/descent route, we add it to the published airways for flight levels 240 through 290, as shown in Figure 5-10, and treat it just as if it were a published airway with traffic in level flight. The velocity distribution would be the same as that for the climb/descent route. To figure the traffic density on the new airway, we divide the number of flight levels conflicting with the average climbing/descending aircraft (in this example, three) by the total number of affected flight levels (six). We multiply this ratio (.5) times the actual density on the climb/descent route to get the density assigned to the new crossing airway added to FL's 240 through 290 as in Figure 5-10. In effect, we are saying that a random aircraft on the climb/descent route will be in potential conflict (ie. within 1000 feet vertically) with three of six flight levels as it crosses airway 3-4, so we use 1/2 the actual climbing/descending traffic density on each of the new airways added to FL 240 through 290. In this way we model the interventions generated by the unpublished climb/descent corridor shown by the dashed line in Figure 5-9.

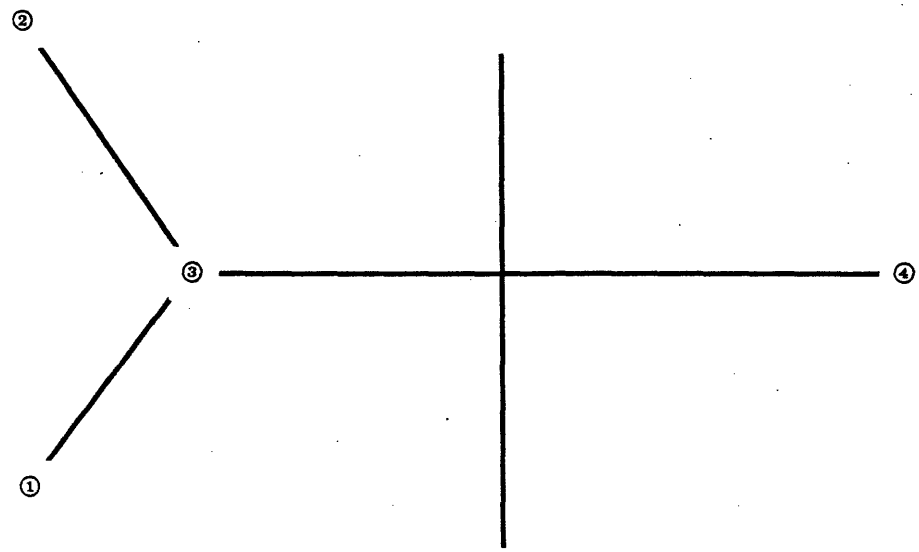


Figure 5-10

To model the effects of those aircraft climbing or descending along a published route, say airway 3-4 in this example, we use essentially the same technique. Using information on climb/descent rates and traffic density (aircraft/NM) in the stream, we assign a certain proportion of the stream to the level flight traffic density along airway 3-4 at each flight level. In Figure 5-11 we see how this additional traffic will be included when computing the overtake rate on the part of airway 3-4 between the source, node 5, and the sink at node 6. A minor technical point: we will also need to consider node 5 to be an intersection of two pseudo-airways of Class II (see Section 2.1.5), to reflect the fact that climbing/descending aircraft modeled to "appear" at node 5 may have an immediate conflict with an aircraft approaching from node 3.

Thus we can reflect the effect of climbing/descending traffic on the total sector intervention rate by suitably modifying the traffic density parameters and network geometries of the individual flight levels, and then summing over the individual intervention rates computed for each modified flight level by the model developed in Chapter 2.

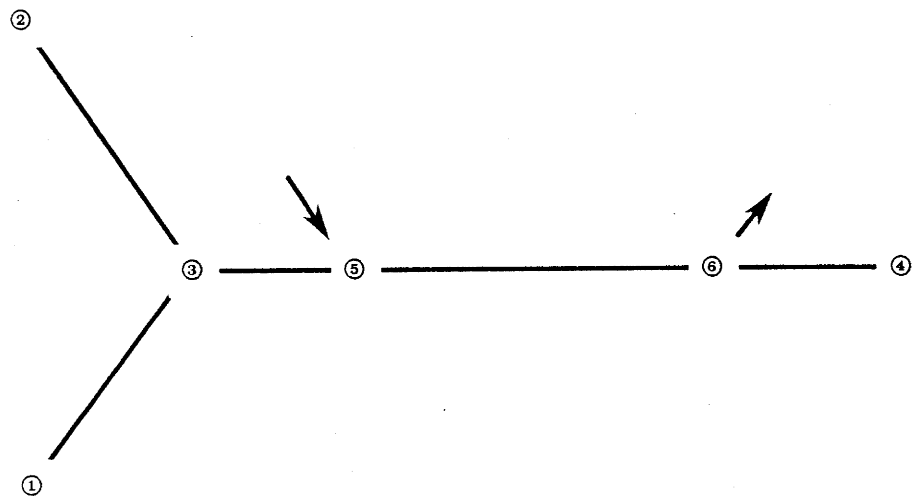


Figure 5-11

Chapter 6
Conclusions and Opportunities for Further Research

6.1 Conclusions

In this thesis we have constructed a generalized model of the domestic en route air traffic control system which can be used to predict the rate at which controllers will need to intervene because of developing conflicts. The model considers both crossing and overtaking conflicts, and includes both on- and off-airway traffic. Further, the model is able to incorporate complex airway intersections, including those involving more than two crossing airways, as well as those which permit aircraft to change airways at the intersection.

The model as described in Chapter 2 is a two-dimensional one, but in Chapter 5 it is generalized to three dimensions by the use of traffic "sources" and "sinks" to represent climbing/descending aircraft which appear and then disappear in successive flight levels. The three-dimensional version was not extended to Terminal Control Areas, due to the decidedly non-random traffic flow into and out of large airports.

In Chapter 3 we used Monte Carlo simulations of simple airway intersections and single-airway-segment overtaking situations (the fundamental building blocks of the model) to validate the expected intervention rates predicted by the analytic model. The simulations produced a useful secondary result, in that they illustrated the often significant variation to be expected about the mean intervention rate. This variation was in turn confirmed by an analytic model which conditioned on the actual (as opposed to expected) traffic density, thus allowing the use of well known equations for the variance of a binomial random variable.

This model, like all others, made certain assumptions about the subject under investigation. Some of the key ones involved the accuracy of our knowledge of aircraft position and velocity, the amount of cross-track deviation along assigned airways, and the degree to which system parameters are in steady state. In Chapter 4 we looked at the sensitivity of the model's predictions to these and other assumptions. We found the model to be relatively insensitive to small, random errors in our knowledge of aircraft position, as well as to symmetrically distributed cross-track deviations about airway centerlines. Our use of a delayed negative exponential interarrival distribution (vice the simple negative exponential) had a more noticeable impact on predicted intervention rates, particularly where overtaking conflicts were concerned. Using a simple negative exponential interarrival distribution (while holding traffic density constant) increased predicted overtaking intervention rates by as much as 50%.

The model appears to be most sensitive to assumptions about steady state behavior. When predictions about controller intervention rates are made over extended periods using aggregated parameters (eg. flow rates averaged over the full period), significant errors can be introduced. These errors are most likely when there are relatively large differences between the long term average flow rate, and the average rates observed during shorter peak periods. These errors are magnified when there is a strong positive or negative correlation between peak traffic flows in various parts of the sector. The errors can be reduced by performing the sector analysis over shorter periods of time, thereby reflecting the changes in mean flow rates.

In summary, these results represent significant extensions of and generalizations to the models currently in the literature. Existing models are limited to small portions of the complex en route traffic network system, most typically single intersections or simple airway segments. These simpler models have not been integrated to represent more complex full-sector networks, nor have they, with few meaningful exceptions, been extended to three dimensions. The gas model has a three-dimensional version, but it assumes that aircraft travel vertically as often,and at the same velocities, as they do horizontally. Geisinger's model allows realistic vertical velocities, but it is limited to small segments of a complete traffic network. Reich's model is truly three-dimensional, but it is limited to oceanic networks comprised of long, parallel airways without significant traffic flow from one airway (or altitude) to another. Finally, all current models predict conflict,not intervention, rates, a less realistic measure of controller workload in a radar-controlled environment.

In this thesis we have also extended beyond an ability to predict expected intervention rates to a discussion of the variance to be expected about the mean rates. We have shown that this variance may be significant, and that over the entire sector it will be a function not only of the variance at each intersection (crossing interventions) and simple airway segment (overtaking interventions), but also of the degree to which there is correlation between the activity levels at each of these constituent parts of the sector.

In Chapter 4 we have shown how this model is sensitive to assumptions about our knowledge of exact aircraft position and velocity, the amount of cross-track variation about an airway centerline, and the aggregation of flow rate parameters. We have shown that paying attention to peak traffic densities by breaking down the length of time over which average intervention rates are predicted, will do more to insure accurate results than a concern for cross-track variations or relatively small errors in airspeed or position data.

6.2 Opportunities for Further Research

6.2.1 Network Development

This model is designed to predict the controller intervention rates in existing air traffic networks, as a function of possibly changing traffic parameters (velocity distributions, flow rates). As such it can be of significant use to FAA management as it tries to predict the effects of such parameter changes on its current network structure.

The use of the model can be extended, however, to include investigations of new networks themselves. We have seen a simple example of this sort of analysis in Chatper 2, where we showed how changes in network geometry can influence intervention rates. Since the model can be implemented to run in real time on a micro-computer, various alternative networks can be evaluated under a variety of traffic loads in order to determine those which might tend to reduce overall intervention rates. The model can also be used to predict changes in intervention rates due to altered procedural rules (eg. a change in minimum separation standards). This sort of case-by-case analysis may be useful under many circumstances; however it may not suffice for generalized investigations into the design (or re-design) of complete, large-scale networks. Generating a methodology which will use the results of this thesis in the construction of optimal air traffic networks (where an "optimal" network minimizes intervention rates) will require substantial additional research.

6.2.2 Optimal Controller Responses to Adverse Weather

Few things disrupt the orderly flow of traffic through an en route sector more than the unexpected appearance of adverse weather. When thunderstorms, icing, or severe turbulence are encountered, controllers often have to re-direct on short notice a substantial portion of their assigned aircraft. Some of the tools discussed in 6.2.1 may be of use here, in what is, after all, a short-term re-design of a (hopefully) near-optimal network gone bad. Both a case-by-case approach (eg. " what to do when a squall line cuts across these two airways") and the application of general principles ("try to keep aircraft segregated according to airspeeds as much as practical") may be of use.

6.2.3 Extension to Terminal Control Areas

In Chapter 5 we noted that the present version of the model, which assumes a fairly randomized distribution of interarrival distances along airways, does not directly apply to most TCA's. The problem, as we have seen, lies in the highly sequenced flow of traffic into and out of most busy commercial airports. Further investigation may illuminate ways to incorporate such non-random flow into the general model. Two possible approaches appear promising. One involves modeling arriving and departing traffic streams as having fixed interarrival distances; another possibility is to consider the interarrival distances to be distributed according to a delayed negative exponential distribution, with the delay now large enough to represent the minimum separation maintained in the arriving/departing streams of traffic. Both techniques may be useful in extending the results of this thesis into Terminal Control Areas.

Appendix

A. The Delayed Negative Exponential Distribution

In this section we will examine some of the properties of the delayed negative exponential distribution (DNE), paying particular attention to how these properties differ from the standard (non-delayed) negative exponential distribution (NE). We will see that some insight can be gained by viewing NE as a special case of DNE.

In Figure A-1 we see a graph of a typical negative exponential distribution. Its probability density function is given by:

$$f_X(x) \quad \begin{aligned} &= 0 && \text{for } x < 0 \\ &= \frac{1}{S}\, exp\left(-\frac{x}{S}\right) && \text{for } x \geq 0 \end{aligned} \qquad (A-1)$$

where S is the expected value of the random variable X.

A delayed negative exponential distribution is very similar, as we see in Figure A-2. The DNE is simply an NE shifted to the right by some positive distance M. Thus the probability density function for the DNE is:

$$f_X(x) \quad \begin{aligned} &= 0 && \text{for } x < M \\ &= \frac{1}{S-M}\, exp\left(-\frac{x}{S-M}\right) && \text{for } x \geq M \end{aligned} \qquad (A-2)$$

We write the pdf in this form because we will see that the expected value of a random variable distributed according to $f_X(x)$ will also be S.

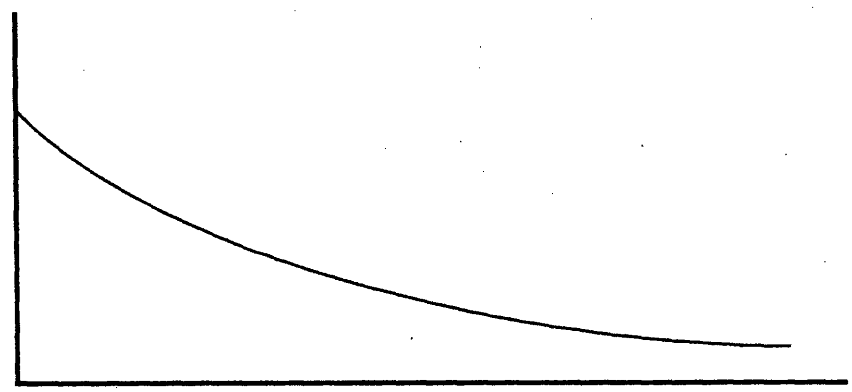


Figure A-1

A Standard (non-delayed) Negative Exponential Distribution

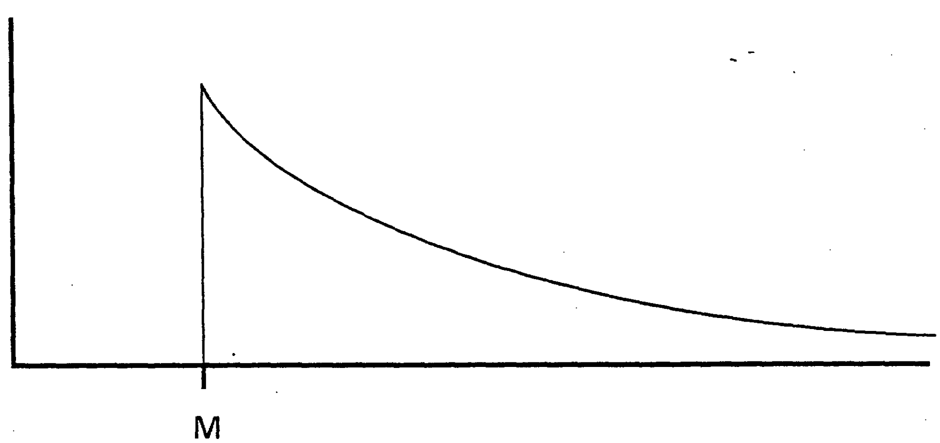


Figure A-2

A Delayed Negative Exponential Distribution with Delay M

One useful way to view the DNE is as follows. Let X be a random variable distributed according to an NE with parameter $1/(S-M)$, and let Y be a random variable distributed according to a DNE with delay M and expectation S. As a shorthand we will write:

$$X \sim NE\,(1/(S-M))$$

$$Y \sim DNE\,(M\,,1/(S-M))$$

Then Y can be viewed as the sum of two random variables

$$Y = M + X \tag{A-3}$$

where M, the delay in the distribution of Y, is a constant (i.e. a random variable with zero variance).

A few simple results follow immediately from (A-3). If we want to compute the expected value of Y, we have

$$\begin{aligned} E[Y] &= E[M] + E[X] \\ &= M + (S-M) \\ &= S\,. \end{aligned} \tag{A-4}$$

Similarly, if we want the variance of Y,

$$\begin{aligned} \sigma_Y^2 &= \sigma_M^2 + \sigma_X^2 \\ &= (S-M)^2 \end{aligned} \tag{A-5}$$

since M and X are independent random variables.

Notice that $Y \sim DNE\,(M\,,1/(S-M))$ and $X \sim NE\,(1/S)$ have the same mean S: while the variance of Y is less than the variance of X (assuming $M > 0$).

From (A-3) we can also compute the LaPlace transform of $f_Y(y)$ to be

$$f_Y^T(\Sigma) = f_M^T(\Sigma) f_X^T(\Sigma)$$

$$= exp(-M\Sigma) \left[\frac{\left(\frac{1}{S-M}\right)}{\left(\frac{1}{S-M}\right) + \Sigma} \right]$$

$$= exp(-M\Sigma) \left[\frac{1}{1 + (S-M)\Sigma} \right] \qquad (A-6)$$

Consider an arrival process with random variable X representing the times between successive arrivals. Let $X \sim NE(1/S)$. One of the key properties of NE is its "memorylessness:" at any time in the arrival stream the distribution of T, the time until the next arrival, is $\sim NE(1/S)$. Further, we know that $E[T] = S$.

If, on the other hand, $X \sim DNE(M, 1/(S-M))$, then the expected time between arrivals is still S, but the process is no longer memoryless: the expected time until the next arrival depends upon how long it has been since the last arrival. Let τ be the time since the last arrival. If $\tau \geq M$, then $T \sim NE(1/(S-M))$ and $E[T] = S - M$. But if $\tau < M$, then $T \sim DNE(M-\tau, 1/(S-M))$ and $E[T] = (M-\tau) + S - M = S - \tau$. The process is no longer memoryless, but it is still conditionally memoryless : given $\tau \geq M$, we have $T \sim NE(1/(S-M))$ and $E[T] = S - M$.

To compute the unconditional expectation of T we can proceed as follows. From Lemma 2 we know that

$$P[\tau < M] = \frac{M}{S}$$

and

$$P[\tau \geq M] = 1 - \frac{M}{S}.$$

We have just seen that

$$E[T \mid \tau \geq M] \;=\; S - M$$

and

$$E[T \mid \tau = \tau_0 < M] \;=\; S - \tau_0 \;;$$

so

$$E[T \mid \tau < M] \;=\; \int_0^M (S - \tau) f_\tau(\tau)\, d\tau \;. \qquad (A-7)$$

Since it seems reasonable to assume that

$$f_\tau(t) \;\begin{cases} = \dfrac{1}{M} & 0 \leq t \leq M \\ = 0 & otherwise \end{cases} \qquad (A-8)$$

we have

$$E[T \mid \tau < M] \;=\; S - \frac{M}{2} \;.$$

Thus

$$E[T] \;=\; E[T \mid \tau < M]\, P[\tau < M] \;+\; E[T \mid \tau \geq M]\, P[\tau \geq M]$$

$$= \left(S - \frac{M}{2}\right)\left(\frac{M}{S}\right) \;+\; \left(S - M\right)\left(1 - \frac{M}{S}\right)$$

$$= S - \left(\frac{2MS - M^2}{2S}\right) \;. \qquad (A-9)$$

While $0 < M \leq S$ we have $E[T] < S$, since the quantity in parenthesis in (2) will be positive. When $M = 0$, we degenerate to an NE(1/S), and $E[T] = S$ as expected. Notice that when $M = S$, $E[T] = S/2$.

We should now take note of the fact that the DNE is in many ways a hybrid, intermediate between two pure "parents." When $M=0$ the DNE is in fact a standard (non-delayed) negative exponential. If, for example, $X \sim$ DNE (0, 1/S), then by the definition of DNE, $X \sim$ NE(1/S) and of course

$$E[X] = S \quad \text{and} \quad \sigma_X^2 = S^2 .$$

Less obvious, perhaps, is the other extreme. As M approaches S we see that

$$\lim_{M \to S} f_X^T(\Sigma) = \lim_{M \to S} \left[exp(-M\Sigma) \ \frac{1}{1 + \Sigma(S - M)} \right]$$

$$= exp(-S\Sigma) \qquad . \qquad (A-10)$$

Thus the other extreme is in fact a distribution with a unit mass at $x=S$. Note that this is consistant with our earlier results:

$$\lim_{M \to S} E[X] = S \qquad (A-11)$$

$$\lim_{M \to S} \sigma_X^2 = 0 \qquad (A-12)$$

Similarly, we found in the random incidence question that

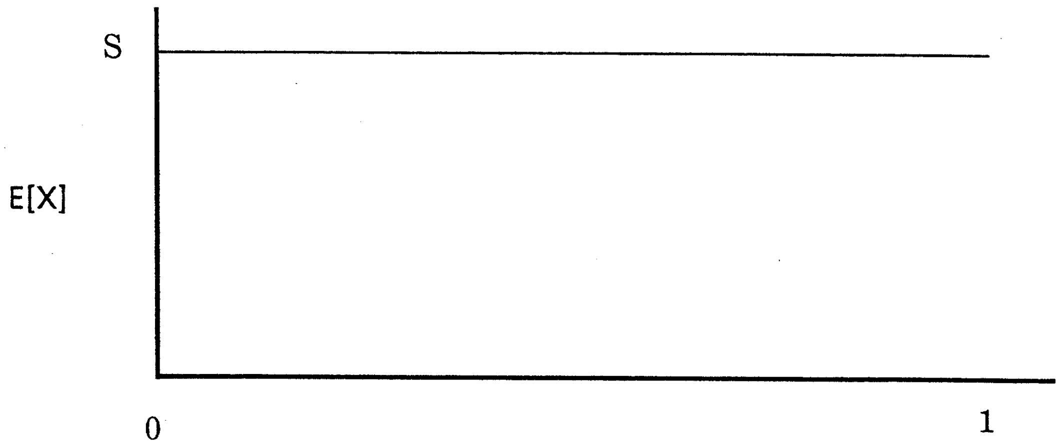
$$\lim_{M \to S} E[T] = S/2$$

and

$$\lim_{M \to S} \sigma_T^2 = S^2/12 ,$$

just as we would expect from a unit point mass distribution.

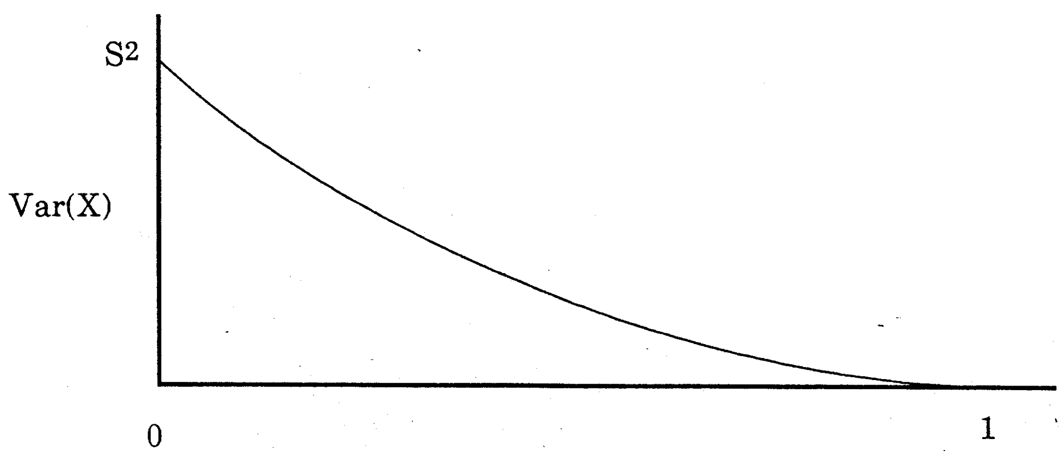
In Figures A-3 we present these results graphically. Remember that $X \sim \text{DNE}(M, 1/(S-M))$, $\alpha \equiv M/S$, and T is the time to the next arrival after a random incidence into the arrival stream.



α

Figure A-3.1

The expectation of X as a function of α



α

Figure A-3.2

The variance of X as a function of α

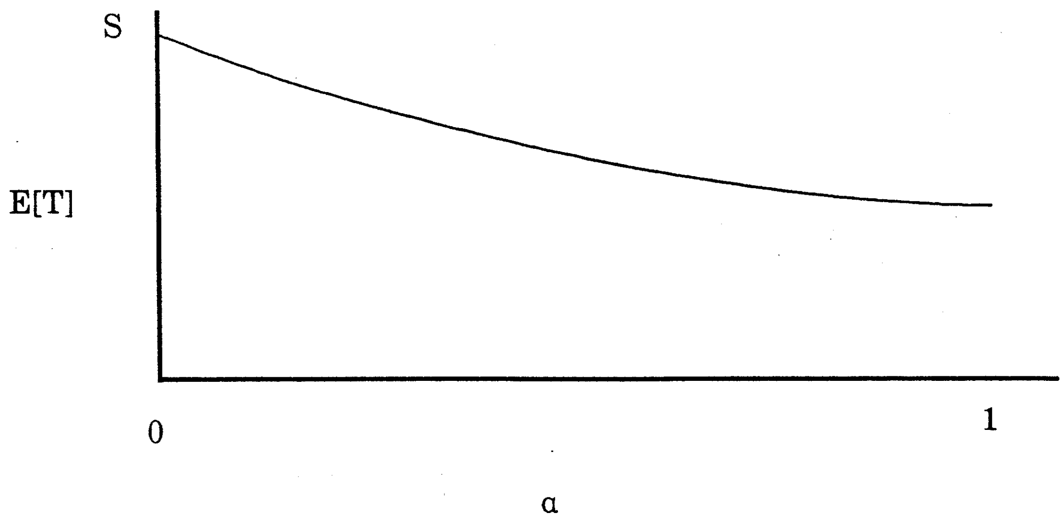


Figure A-3.3

The expectation of the time until the next arrival (T) as a function of α

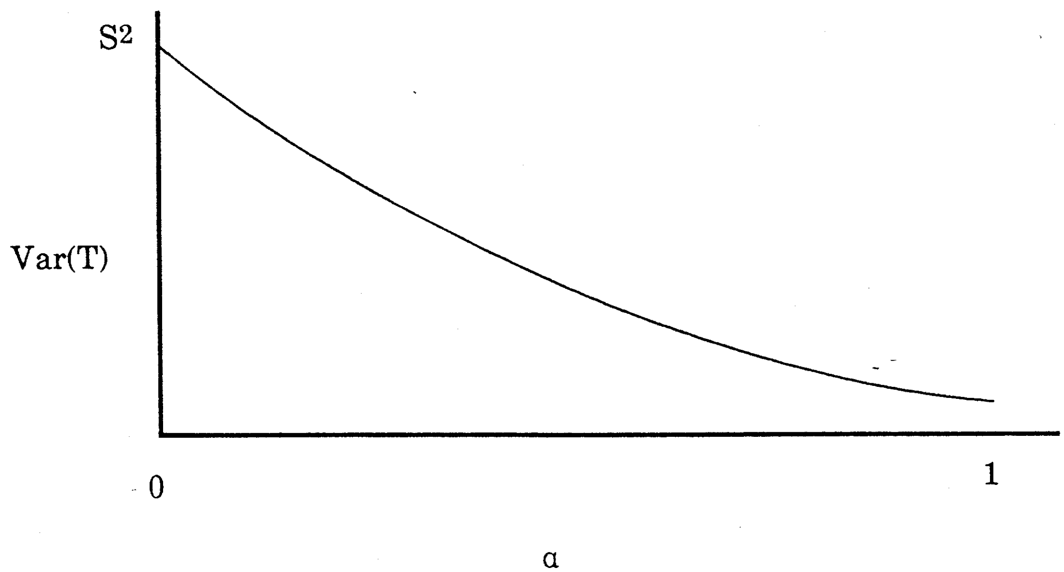


Figure A-3.4

The variance of the time until the next arrival (T) as a function of α

n^{th} Order Arrivals

Another interesting question involves the distribution of the time required for n arrivals. Let X_n be the time between the 0^{th} and the n^{th} arrivals, where each interarrival time is a random variable $\sim$ NE(1/S). Then it is well known that X_n is distributed according to an n^{th} order Erlangian distribution with mean $E[X_n] = nS$ and variance

$$\sigma^2_{X_n} = nS^2 .$$

The pdf for X_n is

$$f_{X_n}(x) \quad \begin{cases} = \dfrac{\left(\dfrac{1}{S}\right)^n x^{n-1} \exp\left(-\dfrac{x}{S}\right)}{(n-1)!} & \text{for } x \geq 0 \\ = 0 & \text{otherwise} \end{cases} \qquad (A-13)$$

and the LaPlace transform of $f_{X_n}(x)$ is

$$f^T_{X_n}(\Sigma) = \left(\frac{1}{1+S\Sigma}\right)^n . \qquad (A-14)$$

Now lets consider the case where each interarrival time is an independent random variable distributed according to DNE(M, (1/(S − M)). Then the time between the 0^{th} and the n^{th} arrivals ($\equiv X_n$) is simply the sum of n independent random variables, and thus

$$f^T_{X_n}(\Sigma) = \left(f^T_X(\Sigma)\right)^n \tag{A-15}$$

where X ~ DNE(M, 1/(S − M)).

This means that

$$f^T_{X_n}(\Sigma) = \left[exp(-M\Sigma)\left(\frac{1}{1-(S-M)\Sigma}\right)\right]^n .$$

$$= exp(-nM\Sigma)\left[\frac{1}{1-(S-M)\Sigma}\right]^n \tag{A-16}$$

This last is simply the LaPlace transform of an n^{th} order Erlangian distribution (with mean equal to n(S − M)) delayed (i.e. shifted to the right) by nM. Thus the n^{th} order arrival time is distributed according to what we might call a delayed n^{th} order Erlangian distribution.

To complete the picture we should note that

$$\lim_{M\to 0} f^T_{X_n}(\Sigma) = \left[\frac{1}{1-S\Sigma}\right]^n \tag{A-17}$$

and

$$\lim_{M\to S} f^T_{X_n}(\Sigma) = exp(-nS\Sigma); \tag{A-18}$$

so the delayed n^{th} order Erlangian is a hybrid, intermediate between a standard n^{th} order Erlangian (when M = 0) and a distribution with a unit point mass at x = nS.

B Geometrical Errors in the Modeling of Aircraft Collsions

Recent attempts to model enroute or terminal area air traffic systems have represented the space occupied by a single aircraft as some regular geometric form in order to simplify the calculation of overlap (i.e. collision) probability. Typically, circles or rectangles are used in two-dimensional models, while cylinders are most common in three dimensions.

It has been claimed that significant over-estimation of collision potential can occur as a result of such simplifications because aircraft profiles are inflated by as much as 100% or more. In this section we will estimate the error generated by representing an aircraft in a two-dimensional collision model by a circle with diameter equal to the aircraft's wingspan. We will see that this error is approximately 10%, a not unreasonable safety margin when one considers the violent aerodynamic interaction that may occur between inflight aircraft passing within a few feet of one another.

The Model

Consider a two-dimensional collision model which represents an aircraft of wingspan W by a circle of diameter W. We will assume that the aircraft measures W from nose to tail as well, so that the "actual" vs. modeled profile is as shown in Figure B-1.

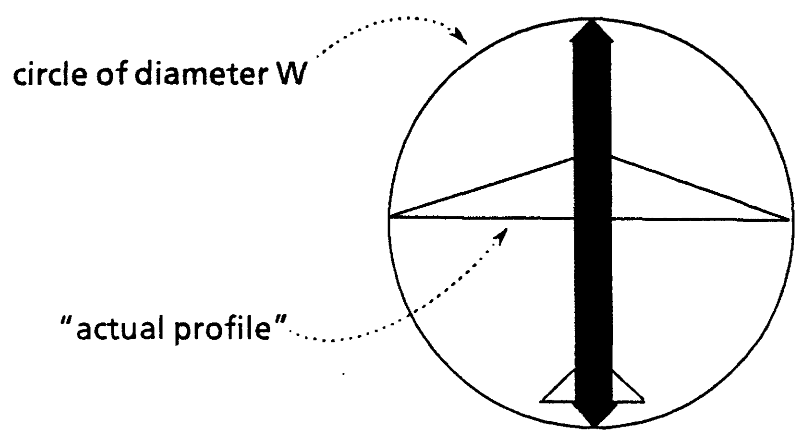


Figure B-1

In the model, a collision is said to occur whenever two adjacent aircraft (i.e. circles) overlap. We wish to determine how often two circles overlap while actual aircraft profiles do not.

In this discussion we will protray a near-miss between aircraft α and β in a frame of reference centered at α. Further, the velocity of β relative to α will always be oriented towards the top of the page. We will measure θ_α and θ_β, the aircraft headings, relative to the top of the page, as in Figure B-2.

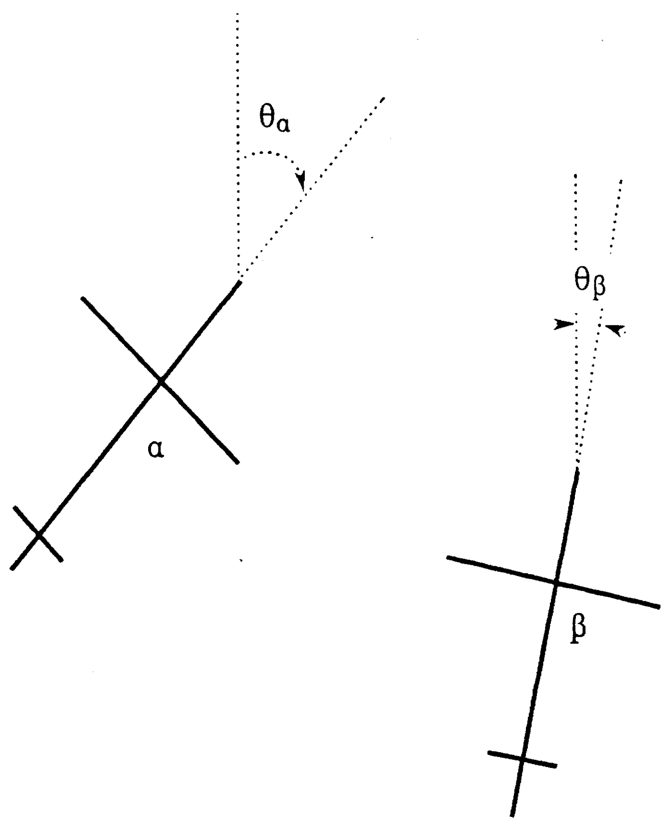


Figure B-2

Clearly the circles centered on α and β will overlap whenever the profiles do. The reverse, however, is not the case, as Figure B-3 shows. Remember, α is stationary and β is moving parallel to the sides of the page. We want to determine how often a situation like the one in Figure B-3 will occur.

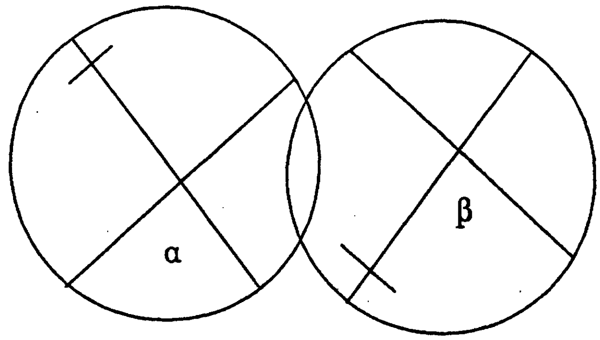


Figure B-3

The key to our argument is the observation that an equivalent representation of the actual profile is a diamond with vertices at the aircraft's nose, tail, and wingtips. This diamond is "equivalent" to the profile in the sense that in a near-miss situation diamonds will overlap almost exactly when profiles do : the diamond model will not over-estimate the collision probability to any significant degree.

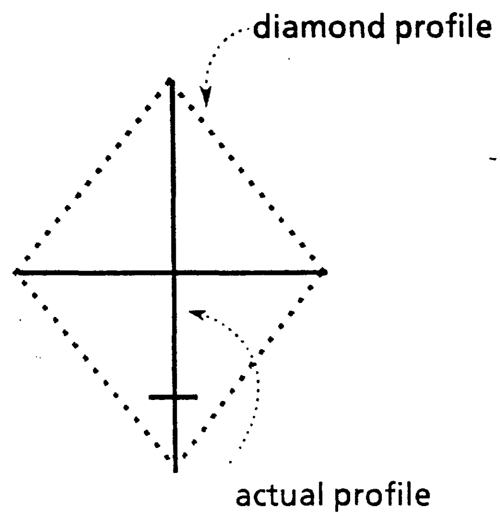


Figure B-4
The Diamond Profile

The equivalence is due to the fact that the diamond and the profile both obstruct the same arc across the horizon when viewed from any point <u>in the plane</u>. This is not the case when one compares profile and circle. See Figure B-5.

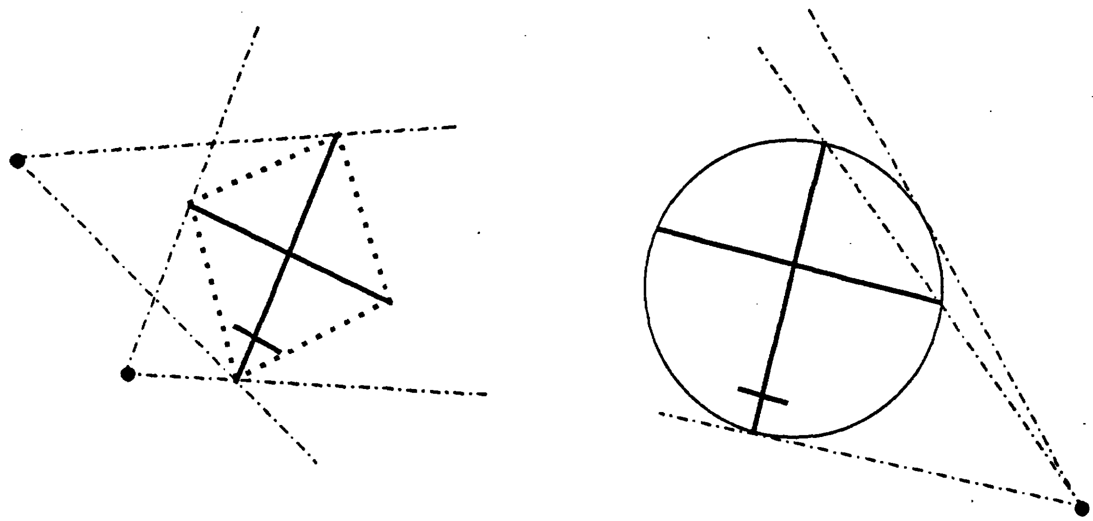


Figure B-5

Thus, to determine the error produced by using a circular aircraft (vs. the profile) we can calculate the error resulting from the circle vs. the diamond. This latter, it turns out, is a fairly straight-forward computation.

Analysis

Consider the situation in Figure B-6, when β passes α on the right (in our frame of reference) and the circles of diameter W are tangent to each other.

It is clear that if we move β's position (as it passes α) to the left up to a distance 2W, an overlap will occur. Further, an overlap will not occur if β moves to the right at all, or to the left more than 2W. Remember, α is stationary and β is moving parallel to the sides of the page,

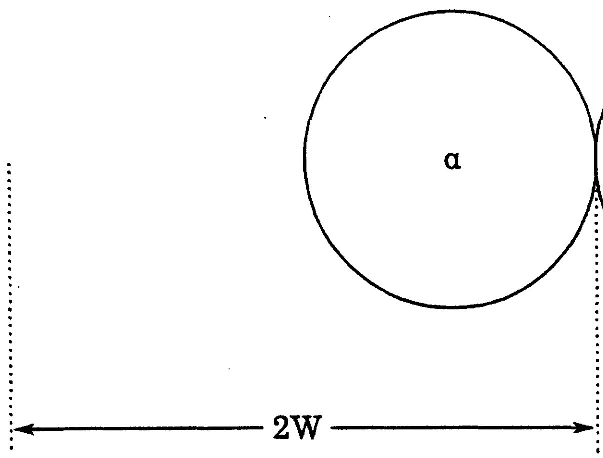


Figure B-6

from bottom to top. Our question now becomes: over what portion of this 2W span will the diamond model fail to register a collision?

The answer to this question depends on θ_α and θ_β, the aircraft headings in our frame of reference. If, as in Figure B-7, $\theta_\alpha = \theta_\beta = 0$ (a simple overtaking situation), there will be no difference at all: if circles overlap, then diamonds will, too. If, on the other hand, $\theta_\alpha = 135°$ and $\theta_\beta = 45°$ as in Figure B-3, the diamond model will register an overlap over a distance smaller than 2W.

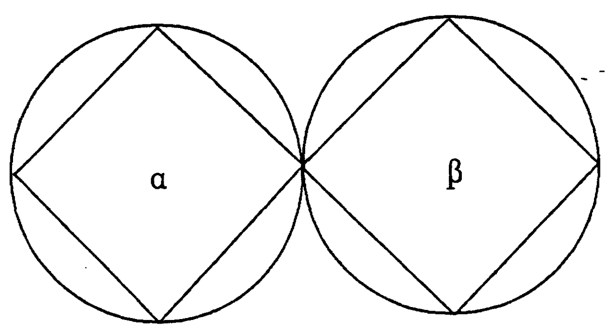


Figure B-7

To see exactly what this distance is, consider a "close-up" of an arbitrary near-miss situation (Figure B-8), where once again β is passing α on the right. As before, the circle model will regiater a collision over 2W, but the diamonds will collide only over $2W - 2x - 2y$ (i.e. they will miss over $x + y$ on either side).

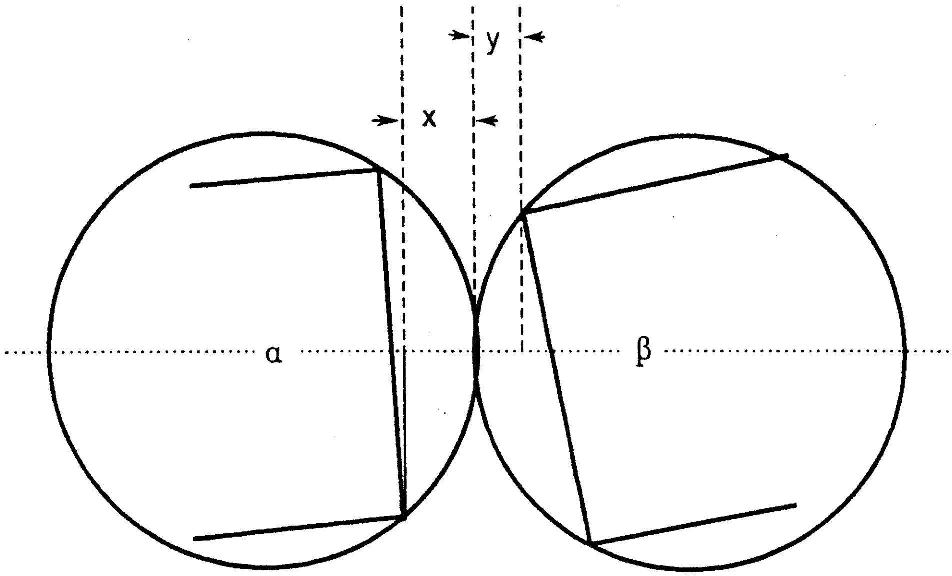


Figure B-8

Now because of the symmetry of the situation, we need only consider values for θ_α and θ_β between 0 and $\pi/4$, so we get

$$x = \frac{W}{2}\left(1 - \cos\theta_\alpha\right)$$

$$y = \frac{W}{2}\left(1 - \cos\theta_\beta\right)$$

If we make the reasonable assumption that all ground tracks over the [0,2W] range are equally likely, we can compute $PE(\theta_\alpha, \theta_\beta)$, the proportion of the time the diamond model "misses" while the circle model collides, to be

$$PE(\theta_\alpha, \theta_\beta) = \frac{2\left(\frac{W}{2}\right)\left[\left(1 - \cos\theta_\alpha\right) + \left(1 - \cos\theta_\beta\right)\right]}{2W} \qquad (B-1)$$

$$= \frac{1}{2}\left(2 - \cos\theta_{\alpha} - \cos\theta_{\beta}\right) \qquad (B-2)$$

A few typical values of PE $(\theta_\alpha, \theta_\beta)$ are shown in Table B-1.

θ_α \ θ_β	0	π/8	π/4
0	0	.038	.146
π/8	.038	.076	.185
π/4	.146	.185	.292

Table B-1

Typical Values of PE $(\theta_\alpha,\theta_\beta)$

Now if we assume that all ordered pairs $(\theta_\alpha, \theta_\beta)$ in $[0,\pi/4] \times [0,\pi/4]$ are equally likely (an assumption we'll discuss in more detail below) then we can calculate an overall probability of error to be

$$PE = \frac{1}{\left(\frac{\pi}{4}\right)^2} \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} PE(\Theta_\alpha, \Theta_\beta)\ d\Theta_\alpha\ d\Theta_\beta$$

$$= \frac{16}{\pi^2} \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{1}{2} (2 - \cos\Theta_\alpha - \cos\Theta_\beta)\, d\Theta_\alpha\, d\Theta_\beta$$

$$= \frac{8}{\pi^2} \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} (2 - \cos\Theta_\alpha - \cos\Theta_\beta)\, d\Theta_\alpha\, d\Theta_\beta$$

$$= .0997 \qquad (B-3)$$

Thus the circle model overestimates collisions between profiles by about 10%.

Now let's return to the assumption that θ_α and θ_β are unifrom over $[0,\pi/4]$, mod $\pi/4$. (By this we mean that the remainder left when θ_α or θ_β is divided by $\pi/4$ is uniform on $[0,\pi/4]$). In a later section an estimate of this distribution is made, with the results shown in Graph B-1. These results indicate that the distribution of θ_α (mod $\pi/4$) is essentially flat, while that of θ_β (mod $\pi/4$) varies by at most 50%. More importantly, the variation is symmetrical for both θ_α and θ_β about $\pi/8$ ($=22.5°$). Thus the value for PE obtained should be reasonably accurate in the general case. In specialized cases where it is not appropriate to assume that θ_α and θ_β are uniformly distributed, the error may be anywhere in the range [0,29.2%], since

$$\min_{\theta_\alpha,\theta_\beta} \{ PE(\theta_\alpha, \theta_\beta) \} = PE(0,0) = 0 \qquad (B-4)$$

while

$$\max_{\theta_\alpha,\theta_\beta} \{ PE(\theta_\alpha, \theta_\beta) \} = PE(\pi/4, \pi/4) = 0.292 \qquad (B-5)$$

In the overtaking case, for example, both θ_α and θ_β will be essentially zero, so the error generated by the circle model will be negligible.

The Error Resulting from the Use of a Square Profile

In some models (e.g. the Reich model of the North Atlantic) a square profile is used to represent the area occupied by the actual aircraft, where the base of the square equals the aircraft wingspan. The square clearly overestimates collision rates even more than the circle in

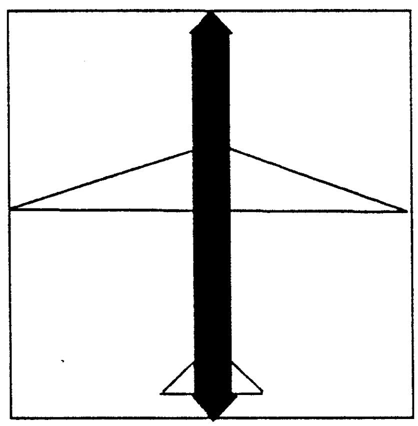


Figure B-9

the general case; in this section a simple calculation will show that overestimate to be about 25% over the circle, and thus about 40% over the actual aircraft profile. Again, these values obtain when θ_α and θ_β are uniformly distributed ; the error in specified cases may be somewhat different.

As we see in Figure B-10, the percent overestimate of the square model over the circle is simply

$$PE(\theta) = \frac{\sqrt{2}\,\frac{W}{2}\,\cos\left(\frac{\pi}{4} - \theta\right)}{\frac{W}{2}}$$

$$= \sqrt{2}\ cos\left(\frac{\pi}{4} - \theta\right) \qquad (B-6)$$

where W is the diameter of the circle.

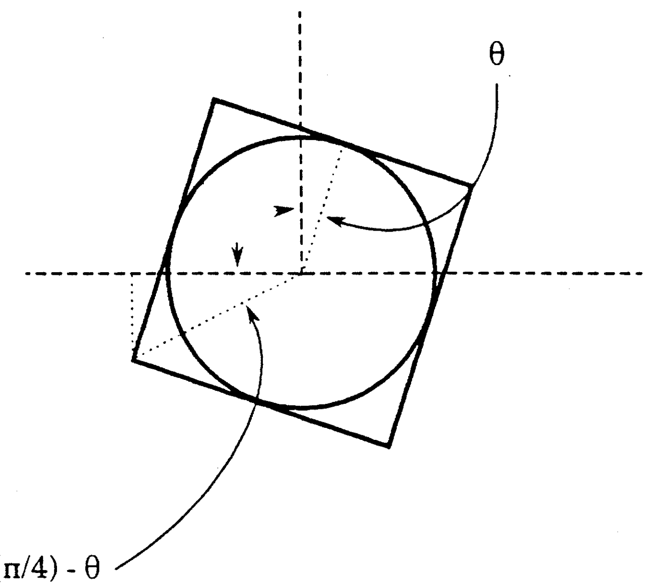


Figure B-10

If we again assume that θ is uniform over $[0,\pi/4]$, we can calculate the expected over-estimate (square vs. circle) as

$$\int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi}\right) PE(\theta)\, d\theta \quad = \quad \frac{4\sqrt{2}}{\pi} \int_0^{\frac{\pi}{4}} cos\left(\frac{\pi}{4} - \theta\right) d\theta$$

$$= \quad \frac{4}{\pi} \qquad (B-7)$$

$$\approx 1.273$$

Using this value we can see that the expected overestimate (square vs. actual aircraft profile) is

$$(1.11)(1.273) = 1.41$$

In specialized cases, when θ_α and θ_β are not uniformly distributed, the overestimate will fall in the range [0, 100%]. As a well known example, consider the Reich model of the North Atlantic air traffic system. Since θ_α and θ_β are tightly distributed around zero (the ground tracks are essentially parallel), the error in the Reich model due to the use of "square aircraft" is essentially zero.

A Counting Algorithm

In this section we describe an algorithm which estimates the distribution of θ_α and θ_β in the frame of reference used to compute PE (that is, in the frame of reference which directs β's velocity relative to α towards the top of the page). We will assume that the magnitude of α's velocity is fixed at 200 knots, while the magnitude of β's velocity can range from 100 to 400 knots.

In Figure B-11 we see the situation under consideration. For a fixed θ_α, we want to determine an estimate of the distribution of "permissible" θ_β's : that is, for what θ_β's is there an airspeed between 100 and 400 knots such that the relative velocity between α and β will be towards the top of the page.

In this algorithm we will "count" angles in increments of 1 degree, and velocities in increments of 1 knot. For each pair $(\theta_\alpha, \theta_\beta)$ we will count as one "occurrence" each velocity v such that the vectors (200 knots, $180 - \theta_\alpha$) and (v, θ_β) sum to a vector directed towards 360°. If rounding is necessary for a given (integral) v, we will round up to the next integral θ_β. We will then count the number of occurrences for each θ_α and θ_β (mod $\pi/4$); the result is shown in Graph B-1.

An example may make this procedure clearer. Let $\theta_\alpha = 30°$ and set $\theta_\beta = 20°$ as in Figure B-12.

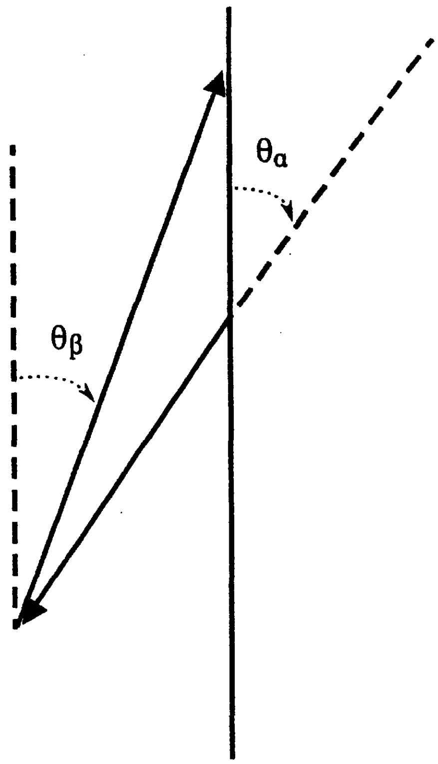


Figure B-11

The only value of v (β's velocity) that will give the disired relative velocity vector in Figure 12 is $v \approx 292.38$ knots. If we reduce θ_β one degree, to $\theta_\beta = 19°$, then v becomes $v \approx 307.155$ knots. Thus we would "count" $(307.155 - 292.38) = 14.775$ occurrences against $\theta_\beta = 20°$ and $\theta_\alpha = 30°$. Notice that we round up to $\theta_\beta = 20°$ when an integral value of v would give $19° < \theta_\beta \leq 20°$.

Following this procedure for each integral θ_α from 1 to 360°, we come up with a count of the relative frequency of each possible integral value of θ_α and θ_β. A graph of this information (mod $\pi/4$) will give an estimate of the relative frequency of θ_α and θ_β. The actual results of this count are presented in Graph B-1.

We make no claim to the absolute accuracy of this discrete counting algorithm. There does seem to be an artifact, for example, which reduces the count at angles very near $n\pi/4$ ($n = 0,1,2, \ldots$). We do claim, however, that the results of this algorithm provide strong support for the assertion that assuming a uniform distribution on θ_α and θ_β does not distort the error estimate to any significant degree.

A flowchart for this algorithm and definitions of associated variables can be found as Flowchart 1 in Appendix C.

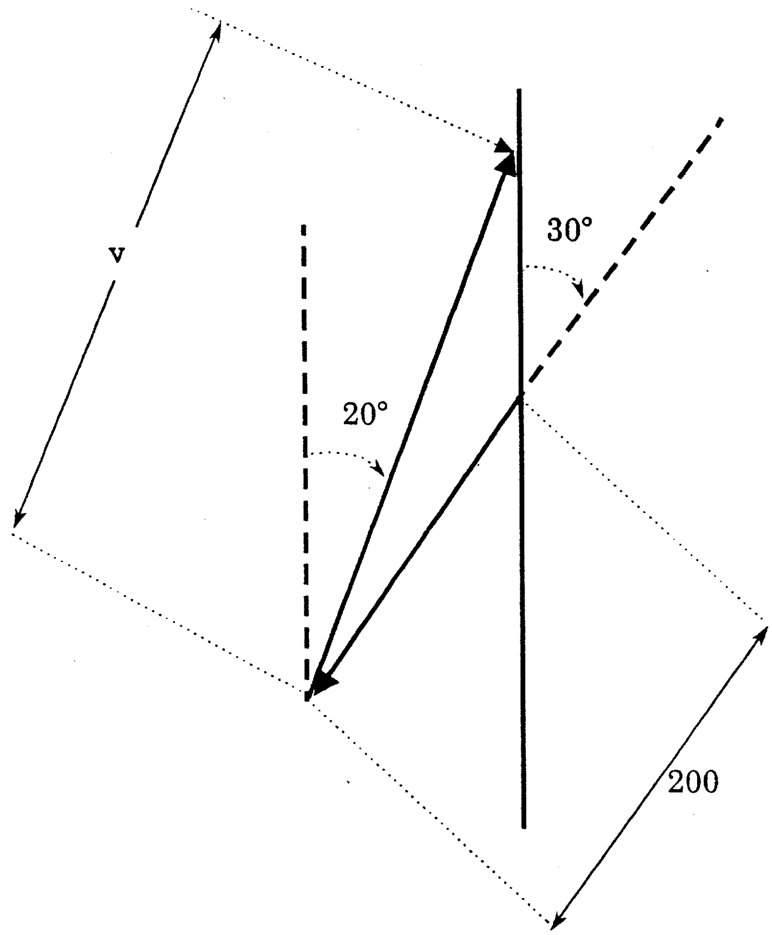


Figure B-12

A Structurally Similar Problem

We have seen that a square 2-dimensional profile overestimates the collision rate of an (inscribed) circular profile by a factor of $4/\pi$. In Larson and Odoni we find that the expected distance between two points in the plane under a Manhattan metric exceeds the expected distance under a Euclidean metric by a factor of $4/\pi$ as well. Is this simply a coincidence?

It is not. Consider a circle C of diameter W. Let r be a random line segment from the center of the circle to a point on the circumference. Let θ be the clock-wise angle between the 45° diagonal and r as shown in Figure B-13.

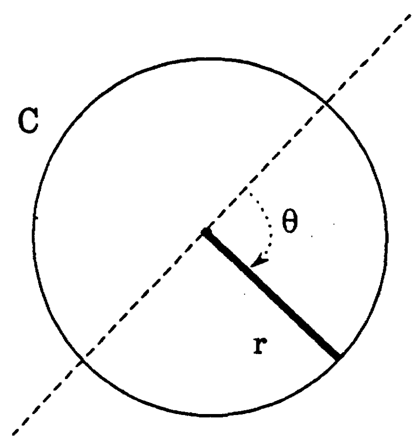


Figure B-13

Now circumscribe about C a square S, one of whose diagonals contains r. We now have the situation as shown in Figure B-14.

By symmetry we can restrict θ to $[0, \pi/4]$.

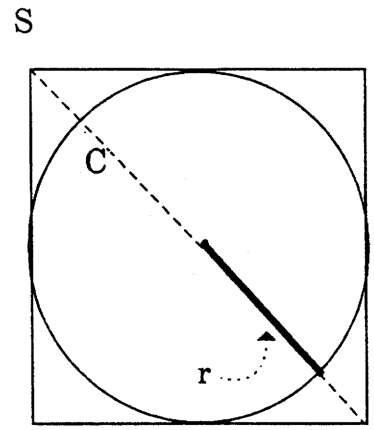


Figure B-14

In the main body of this section we were interested in comparing the length of r (which is W/2) to the length of the horizontal component of the half-diagonal of S through r. See Figure B-15.

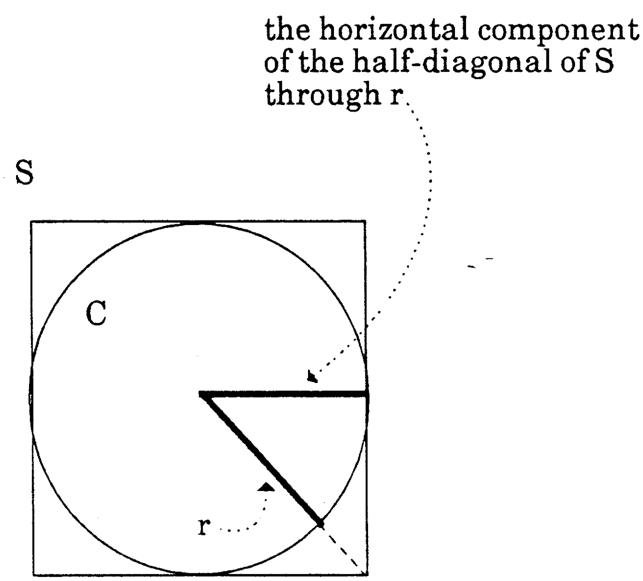


Figure B-15

We showed that the ratio between the length of the horizontal component of the half-diagonal containing r to the radius of C was just

$$PE(\theta) = \sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right). \qquad (B-8)$$

Using the fact that

$$\cos(A + B) = \cos A \cos B - \sin A \sin B , \qquad (B-9)$$

we get

$$PE(\theta) = \sqrt{2}\left[\cos\frac{\pi}{4}\cos(-\theta) - \sin\frac{\pi}{4}\sin(-\theta)\right]$$

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right]$$

$$= \cos\theta + \sin\theta . \qquad (B-10)$$

But $\cos\theta + \sin\theta$ is just the ratio of the sum of the lengths of the opposite and adjacent sides to the hypotenuse, of a right triangle having an angle θ and a unit hypotenuse (see Figure B-15). This, in turn, is the ratio of Manhattan to Euclidean travel distances between two points which define a line at angle θ with the horizontal axis.

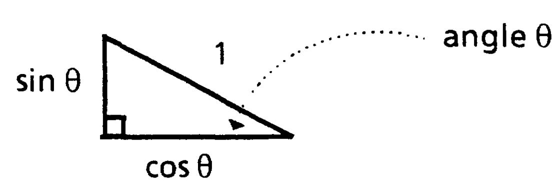


Figure B-15

If we assume a uniform distribution of θ over $[0, 2\pi]$, then

$$E[PE(\theta)] = \frac{4}{\pi} \qquad (B-11)$$

is both the expected ratio of Manhattan to Euclidean travel distances (from Larson and Odoni), as well as the average overestimate of collision rate for square vs. circular aircraft profiles in the plane.

This suggests another way to derive Larson and Odoni's result. Instead of holding the two points in the plane fixed and integrating over all possible x-axis directions, one might hold the coordinate axes and one of the points (call it p) fixed, and integrate over all points r units away from p (i.e. over the circle of radius r centered at p). Then

$$E[PE(\theta)] = \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \cos\theta + \sin\theta \; d\theta$$

$$= \frac{4}{\pi} \left[\sin\theta - \cos\theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{4}{\pi} \left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - 0 + 1 \right]$$

$$= \frac{4}{\pi} \qquad (B-12)$$

Since the result is independent of r, it will hold for all pairs of points in the plane, thus agreeing with Larson and Odoni's derivation.

C Programs and Flowcharts

VARIABLES FOR FLOWCHART 1;

A : $180 - \theta_\alpha$

B : θ_β

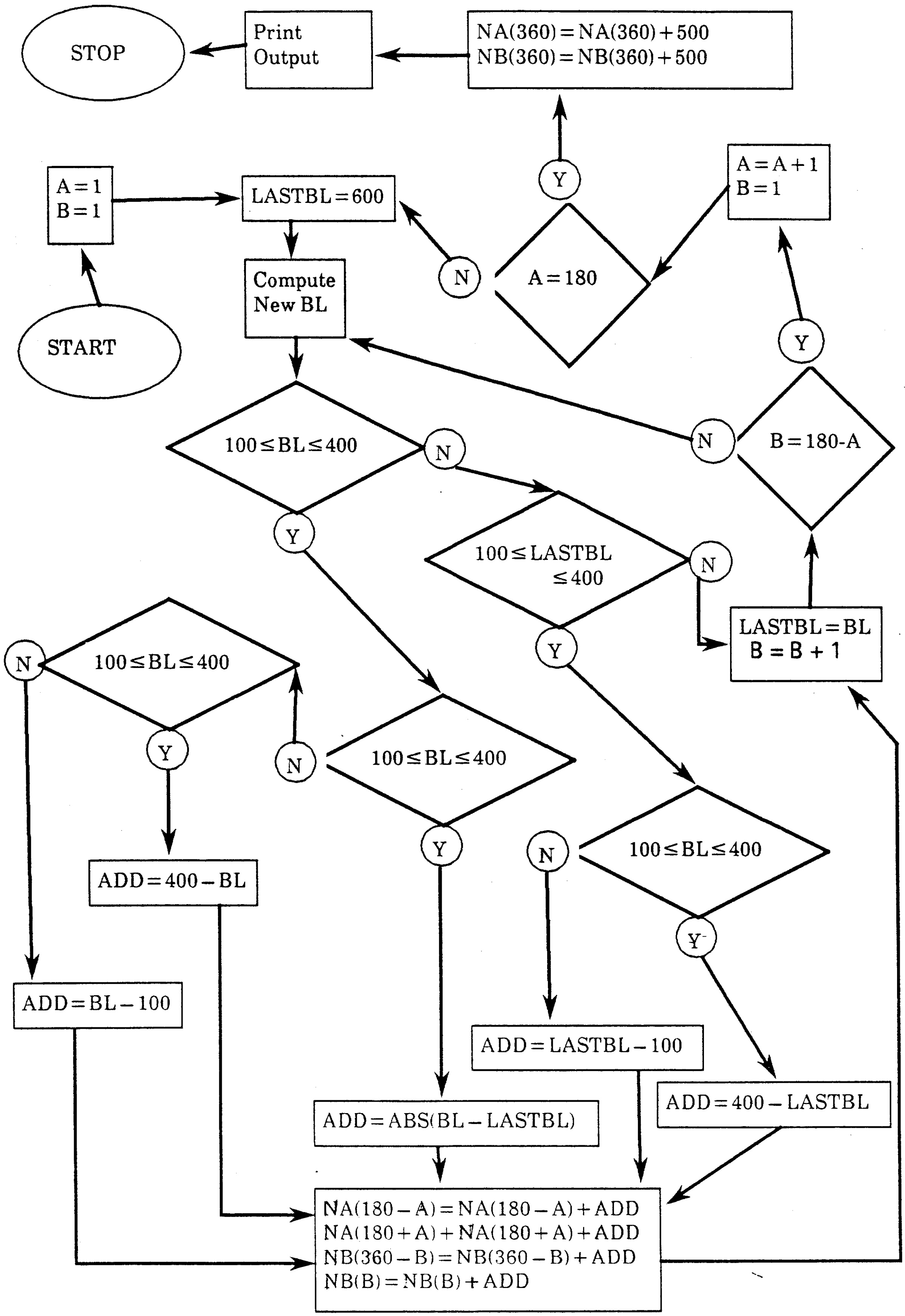
BL : magnitude of velocity of aircraft β

LASTBL : magnitude of velocity of aircraft β in last iteration

ADD : the number of occurrences in this iteration

NA(X) : the number of occurrences counted to date for $\theta_\alpha = X$

NB(X) : the number of occurrences counted to date for $\theta_\beta = X$



Flowchart 1
A Counting Algorithm

PROGRAM 2

```
10 ' PROG6.BAS
20 '
30 '
40 '
50 ' THIS PROGRAM COMPUTES CONTROLLER INTERVENTION RATE DUE TO CROSSING
60 ' AND OVERTAKING
70 ' CONFLICTS FOR THE TYPICAL TWO-DIMENSIONAL SECTOR IN CHAPTER 1.
80 '
90 '    THIS INCLUDES BOTH INTERSECTIONS.  CORRECTS ERRORS IN PROG4,
100 '          AND RENUMBERS.
110 '
120 '
130 ' INPUT TRAFFIC DENSITY, AIRSPEEDS, AND AIRSPEED DISTRIBUTIONS
140 '
150 DIM ASN(6,8), VEL(6,8,20), LAM(6,8), PMF(6,8,20)
160 PI = 3.14159/180
170 CLS
180 INPUT "WHAT IS YOUR MINIMUM SEPARATION?  "; M
190 '
200 PRINT
210 INPUT "HOW MANY AIRSPEEDS ALONG ARC 1-3  "; ASN(1,3)
220 INPUT "HOW MANY AIRSPEEDS ALONG ARC 2-3  "; ASN(2,3)
230 INPUT "HOW MANY AIRSPEEDS ALONG ARC 4-6  " ; ASN(4,6)
240 INPUT "HOW MANY AIRSPEEDS ALONG ARC 5-6  " ; ASN(5,6)
250 '
260 PRINT
270 PRINT "FROM SLOWEST TO FASTEST, LIST THE AIRSPEEDS ON ARC 1-3 :"
280 FOR Q1 = 1 TO ASN(1,3)
290 INPUT VEL(1,3,Q1)
300 NEXT Q1
310 PRINT "NOW DO THE SAME FOR ARC 2-3:"
320 FOR Q1 = 1 TO ASN(2,3)
330 INPUT VEL(2,3,Q1)
340 NEXT Q1
350 PRINT "AND ARC 4-6: "
360 FOR Q1 = 1 TO ASN(4,6)
370 INPUT VEL(4,6,Q1)
380 NEXT Q1
390 PRINT "AND ARC 5-6; "
400 FOR Q1 = 1 TO ASN(5,6)
410 INPUT VEL(5,6,Q1)
420 NEXT Q1
430 '
440 PRINT
450 PRINT "IN ORDER OF INCREASING AIRSPEEDS, "
460 PRINT "INPUT THE PMF VALUES FOR EACH AIRSPEED ALONG ARC 1-3: "
470 FOR Q1 = 1 TO ASN(1,3)
480 INPUT PMF(1,3,Q1)
490 NEXT Q1
500 PRINT "NOW DO THE SAME FOR ARC 2-3"
```

```
510 FOR Q1 = 1 TO ASN(2,3)
520 INPUT PMF(2,3,Q1)
530 NEXT Q1
540 PRINT "AND ARC 4-6: "
550 FOR Q1 = 1 TO ASN(4,6)
560 INPUT PMF(4,6,Q1)
570 NEXT Q1
580 PRINT "AND ARC 5-6: "
590 FOR Q1 = 1 TO ASN(5,6)
600 INPUT PMF(5,6,Q1)
610 NEXT Q1
620 '
630 PRINT
640 INPUT  "WHAT IS THE TRAFFIC DENSITY (A/C PER HOUR) ON ARC 1-3? ";LAM(1,3)
650 INPUT "ARC 2-3? "; LAM(2,3)
660 INPUT "ARC 4-6? "; LAM(4,6)
670 INPUT "ARC 5-6? "; LAM(5,6)
680 '
690 ASN(3,6) = ASN(1,3) + ASN(2,3)
700 FOR Q1 = 1 TO ASN(1,3)
710 VEL(3,6,Q1) = VEL(1,3,Q1)
720 NEXT Q1
730 FOR Q1 = 1 TO ASN(2,3)
740 VEL(3,6,Q1+ASN(1,3)) = VEL(2,3,Q1)
750 NEXT Q1
760 '
770 FOR Q1 = 1 TO ASN(1,3)
780 PMF(3,6,Q1) = PMF(1,3,Q1)*LAM(1,3)/(LAM(1,3)+LAM(2,3))
790 NEXT Q1
800 FOR Q1 = 1 TO ASN(2,3)
810 PMF(3,6,Q1+ASN(1,3)) = PMF(2,3,Q1)*LAM(2,3)/(LAM(1,3)+LAM(2,3))
820 NEXT Q1
830 '
840 LAM(3,6) = LAM(1,3) + LAM(2,3)
850 '
860 INPUT "WHAT FRACTION OF 4-6 TRAFFIC EXITS ON 6-7  " ; PART4667
870 INPUT "WHAT FRACTION OF 3-6 TRAFFIC  " ; PART3667
880 INPUT "AND 5-6  " ; PART5667
890 RC = 0 : RC3 = 0 : RC6 = 0 :  '     SET RC TO ZERO
900 '
910 '
920 ' INTERSECTION 3
930 '
940 ' COMPUTE RC FOR TRAFFIC ON ARC 1-3 CROSSING ARC 2-3
950 '
960 ALF =3.14159* 95/180  :  BET =3.14159* 50/180  :  GAM = 0
970 FOR Q1 = 1 TO ASN(1,3)
980 FOR Q2 = 1 TO ASN(2,3)
990 V1 = VEL(1,3,Q1)  :  V2 = VEL(2,3,Q2) : D = 0
1000 SEXP =( V2/LAM(2,3) )/PMF(2,3,Q2)
```

```
1010 GOSUB 1520
1020 RC3 = RC3 + PCON*LAM(1,3)*PMF(1,3,Q1)
1030 NEXT Q2
1040 NEXT Q1
1050 '
1060 '
1070 ' COMPUTE RC FOR TRAFFIC ON ARC 2-3 CROSSING ARC 1-3
1080 '
1090 ALF =3.14159* 95/180  :  BET =3.14159* 45/180  :  GAM = 0
1100 FOR Q1 = 1 TO ASN(2,3)
1110 FOR Q2 = 1 TO ASN(1,3)
1120 V1 = VEL(2,3,Q1)  :  V2 = VEL(1,3,Q2) : D = 0
1130 SEXP = (V2/LAM(1,3))/PMF(1,3,Q2)
1140 GOSUB 1520
1150 RC3 = RC3 + PCON*LAM(2,3)*PMF(2,3,Q1)
1160 NEXT Q2
1170 NEXT Q1
1180 '
1190 '
1200 PRINT
1210 PRINT "RC3 IS  "; RC3
1220 GOTO 1930
1230 '
1240 '
1250 '
1260 '
1270 ' CLASS I SUBROUTINE
1280 '
1290 K = V2/V1
1300 ABET = K - COS(BET)
1310 IF K=1 AND BET=0 THEN C=1 : GOTO 1440 ELSE KBET=(K^2 +1 -2*K*COS(BET))^(-1)
1320 AGAM = K - COS(GAM)
1330 KGAM = (K^2 +1 - 2*K*COS(GAM))^(-1)
1340 IF K=1 THEN 1360
1350 SALF = ((-AALF*KALF)^2 + (1 - K*AALF*KALF)^2 - 2*(-AALF*KALF)*(1 -

              K*AALF*KALF)*COS(ALF))^(.5)
1360 SBET = ((-ABET*KBET)^2 + (1 - K*ABET*KBET)^2 - 2*(-ABET*KBET)*(1 -

              K*ABET*KBET)*COS(BET))^(.5)
1370 SGAM = ((-AGAM*KGAM)^2 + (1 - K*AGAM*KGAM)^2 - 2*(-AGAM*KGAM)*(1 -

              K*AGAM*KGAM)*COS(GAM))^(.5)
1380 IF K=1 THEN C=SBET^(-1) : GOTO 1440
1390 IF ABET*KBET < 0 THEN D2 = 1 ELSE

        IF V1/V2 >= ABET*KBET AND ABET*KBET >= 0 THEN D2 = SBET ELSE D2 = V1/V2
1400 IF V1/V2 >= AGAM*KGAM THEN D3 = V1/V2 ELSE D3 = SGAM
1410 IF D2 <= D3 THEN D = D2 ELSE D = D3
1420 C = D^(-1)
1430 '
1440 IF C < 1 THEN PCON = 0  ELSE PCON = 1 - EXP((M - C*M)/(SEXP - M))
1450 '
1460 RETURN
1470 '
1480 '
1490 '
1500 ' CLASS II SUBROUTINE
```

```
1510 '
1520 K = V2/V1
1530 AALF = K - COS(ALF)
1540 KALF = (K^2 + 1 - 2*K*COS(ALF))^(-1)
1550 ABET = K - COS(BET)
1560 IF K=1 AND BET=0 THEN C=1:GOTO 1650 ELSE KBET =(K^2 +1 - 2*K*COS(BET))^(-1)
1570 SALF = ((-AALF*KALF)^2 + (1 - K*AALF*KALF)^2 - 2*(-AALF*KALF)*(1 -
            K*AALF*KALF)*COS(ALF))^(.5)
1580 SBET = ((-ABET*KBET)^2 + (1 - K*ABET*KBET)^2 - 2*(-ABET*KBET)*(1 -
            K*ABET*KBET)*COS(BET))^(.5)   : IF K=1 THEN 1590
1590 IF K=1 THEN C=SBET^(-1) : GOTO 1650
1600 IF KALF*AALF >= 0 THEN D1 = 1 ELSE D1 = SALF
1610 IF ABET*KBET < 0 THEN D2 = 1 ELSE
        IF V1/V2 >= ABET*KBET AND ABET*KBET >= 0 THEN
            D2 = SBET ELSE D2 = V1/V2
1620 IF D1 <= D2 THEN D = D1 ELSE D = D2
1630 C = D^(-1)
1640 '
1650 PCON = 1-((SEXP-M)/SEXP)*EXP((M-C*M)/(SEXP-M))
1660 '
1670 RETURN
1680 '
1690 ' CLASS III SUBROUTINE
1700 '
1710 K = V2/V1
1720 AALF = K - COS(ALF)
1730 KALF = (K^2 + 1 - 2*K*COS(ALF))^(-1)
1740 ABET = K - COS(BET)
1750 IF K=1 AND BET=0 THEN C=1: GOTO 1900 ELSE KBET =(K^2 +1 -2*K*COS(BET))^(-1)
1760 AGAM = K - COS(GAM)
1770 KGAM = (K^2 +1 - 2*K*COS(GAM))^(-1)
1780 SALF = ((-AALF*KALF)^2 + (1 - K*AALF*KALF)^2 - 2*(-AALF*KALF)*(1 -
            K*AALF*KALF)*COS(ALF))^(.5)
1790 SBET = ((-ABET*KBET)^2 + (1 - K*ABET*KBET)^2 - 2*(-ABET*KBET)*(1 -
            K*ABET*KBET)*COS(BET))^(.5)
1800 SGAM = ((-AGAM*KGAM)^2 + (1 - K*AGAM*KGAM)^2 - 2*(-AGAM*KGAM)*(1 -
            K*AGAM*KGAM)*COS(GAM))^(.5)
1810 '
1820 IF K = 1  THEN C = SBET^(-1) : GOTO 1900
1830 IF AALF*KALF > 0 THEN D1 = 1 ELSE D1 = SALF
1840 IF ABET*KBET < 0 THEN D2 = 1 ELSE IF (V1/V2 >= ABET*KBET AND ABET*KBET)=0)
        THEN D2 = SBET ELSE D2 = V1/V2
1850 IF V1/V2 >= AGAM*KGAM THEN D3 = V1/V2 ELSE D3 = SGAM
1860 IF D1 <= D2 AND D1 <= D3 THEN D = D1 ELSE IF D2 <= D3 THEN D=D2 ELSE D = D3
1870 C = D^(-1)
1880 '
1890 '
1900 PCON = 1-((SEXP-M)/SEXP)*EXP((M-C*M)/(SEXP-M))
```

```
1910 '
1920 RETURN
1930 '
1940 '      INTERSECTION 6
1950 '
1960 '
1970 '         PSEUDOAIRWAYS 3-6-7 AND 3-6-8
1980 '
1990 ALF = 0         : BET =  45*PI      : GAM = 105*PI
2000 FOR Q1 = 1 TO ASN(3,6)
2010 FOR Q2 = 1 TO ASN(3,6)
2020 V1 = VEL(3,6,Q1) : V2 = VEL(3,6,Q2)
2030 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(1-PART3667))
2040 GOSUB 1270
2050 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*PART3667
2060 NEXT Q2
2070 NEXT Q1
2080 '
2090 BET =  60*PI
2100 FOR Q1 = 1 TO ASN(3,6)
2110 FOR Q2 = 1 TO ASN(3,6)
2120 V1 = VEL(3,6,Q1) : V2 = VEL(3,6,Q2)
2130 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(PART3667))
2140 GOSUB 1270
2150 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*(1-PART3667)
2160 NEXT Q2
2170 NEXT Q1
2180 '
2190 '          PSEUDOAIRWAYS 3-6-7 AND 4-6-7
2200 '
2210 ALF = PI*35    : BET = PI*45      : GAM = PI*0
2220 FOR Q1 = 1 TO ASN(3,6)
2230 FOR Q2 = 1 TO ASN(4,6)
2240 V1 = VEL(3,6,Q1) : V2 = VEL(4,6,Q2)
2250 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(PART4667))
2260 GOSUB 1500
2270 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*(PART3667)
2280 NEXT Q2
2290 NEXT Q1
2300 '
2310 BET =  80*PI
2320 FOR Q1 = 1 TO ASN(4,6)
2330 FOR Q2 = 1 TO ASN(3,6)
2340 V1 = VEL(4,6,Q1) : V2 = VEL(3,6,Q2)
2350 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(PART3667))
2360 GOSUB 1500
2370 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(PART4667)
2380 NEXT Q2
2390 NEXT Q1
2400 '
```

```
2410 '        PSEUDOAIRWAYS 3-6-7 AND 4-6-8
2420 '
2430 ALF = 35*PI   : BET =  60*PI     : GAM = 105*PI
2440 FOR Q1 = 1 TO ASN(3,6)
2450 FOR Q2 = 1 TO ASN(4,6)
2460 V1 = VEL(3,6,Q1) : V2 = VEL(4,6,Q2)
2470 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(1-PART4667))
2480 GOSUB 1690
2490 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*PART3667
2500 NEXT Q2
2510 NEXT Q1
2520 '
2530 BET =  60*PI
2540 FOR Q1 = 1 TO ASN(4,6)
2550 FOR Q2 = 1 TO ASN(3,6)
2560 V1 = VEL(4,6,Q1) : V2 = VEL(3,6,Q2)
2570 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(PART3667))
2580 GOSUB 1690
2590 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(1-PART4667)
2600 NEXT Q2
2610 NEXT Q1
2620 '
2630 '        PSEUDOAIRWAYS 3-6-7 AND 5-6-7
2640 '
2650 ALF = 60*PI   : BET =  15*PI     : GAM = 0
2660 FOR Q1 = 1 TO ASN(3,6)
2670 FOR Q2 = 1 TO ASN(5,6)
2680 V1 = VEL(3,6,Q1) : V2 = VEL(5,6,Q2)
2690 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(PART5667))
2700 GOSUB 1500
2710 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*PART3667
2720 NEXT Q2
2730 NEXT Q1
2740 '
2750 BET =  45*PI
2760 FOR Q1 = 1 TO ASN(5,6)
2770 FOR Q2 = 1 TO ASN(3,6)
2780 V1 = VEL(5,6,Q1) : V2 = VEL(3,6,Q2)
2790 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(PART3667))
2800 GOSUB 1500
2810 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(PART5667)
2820 NEXT Q2
2830 NEXT Q1
2840 '
2850 '        PSEUDOAIRWAYS 3-6-7 AND 5-6-8
2860 '
2870 ALF = 60*PI   : BET =  15*PI     : GAM = 105*PI
2880 FOR Q1 = 1 TO ASN(3,6)
2890 FOR Q2 = 1 TO ASN(5,6)
2900 V1 = VEL(3,6,Q1) : V2 = VEL(5,6,Q2)
```

```
2910 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(1-PART5667))
2920 GOSUB 1690
2930 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*PART3667
2940 NEXT Q2
2950 NEXT Q1
2960 '
2970 BET =  60*PI
2980 FOR Q1 = 1 TO ASN(5,6)
2990 FOR Q2 = 1 TO ASN(3,6)
3000 V1 = VEL(5,6,Q1) : V2 = VEL(3,6,Q2)
3010 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(PART3667))
3020 GOSUB 1690
3030 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(1-PART5667)
3040 NEXT Q2
3050 NEXT Q1
3060 '
3070 '        PSEUDOAIRWAYS 3-6-8 AND 4-6-7
3080 '
3090 ALF = 35*PI   : BET =  25*PI     : GAM = 105*PI
3100 FOR Q1 = 1 TO ASN(3,6)
3110 FOR Q2 = 1 TO ASN(4,6)
3120 V1 = VEL(3,6,Q1) : V2 = VEL(4,6,Q2)
3130 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(PART4667))
3140 GOSUB 1690
3150 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*(1-PART3667)
3160 NEXT Q2
3170 NEXT Q1
3180 '
3190 BET =  45*PI
3200 FOR Q1 = 1 TO ASN(4,6)
3210 FOR Q2 = 1 TO ASN(3,6)
3220 V1 = VEL(4,6,Q1) : V2 = VEL(3,6,Q2)
3230 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(1-PART3667))
3240 GOSUB 1690
3250 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(PART4667)
3260 NEXT Q2
3270 NEXT Q1
3280 '
3290 '        PSEUDOAIRWAYS 3-6-8 AND 4-6-8
3300 '
3310 ALF = 35*PI   : BET =  25*PI     : GAM = 0*PI
3320 FOR Q1 = 1 TO ASN(3,6)
3330 FOR Q2 = 1 TO ASN(4,6)
3340 V1 = VEL(3,6,Q1) : V2 = VEL(4,6,Q2)
3350 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(1-PART4667))
3360 GOSUB 1500
3370 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*(1-PART3667)
3380 NEXT Q2
3390 NEXT Q1
3400 '
```

```
3410 BET =  60*PI
3420 FOR Q1 = 1 TO ASN(4,6)
3430 FOR Q2 = 1 TO ASN(3,6)
3440 V1 = VEL(4,6,Q1) : V2 = VEL(3,6,Q2)
3450 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(1-PART3667))
3460 GOSUB 1500
3470 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(1-PART4667)
3480 NEXT Q2
3490 NEXT Q1
3500 '
3510 '          PSEUDOAIRWAYS 3-6-8 AND 5-6-7
3520 '
3530 ALF = 60*PI    : BET = 120*PI      : GAM = PI*105
3540 FOR Q1 = 1 TO ASN(3,6)
3550 FOR Q2 = 1 TO ASN(5,6)
3560 V1 = VEL(3,6,Q1) : V2 = VEL(5,6,Q2)
3570 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(PART5667))
3580 GOSUB 1690
3590 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*(1-PART3667)
3600 NEXT Q2
3610 NEXT Q1
3620 '
3630 BET =  45*PI
3640 FOR Q1 = 1 TO ASN(5,6)
3650 FOR Q2 = 1 TO ASN(3,6)
3660 V1 = VEL(5,6,Q1) : V2 = VEL(3,6,Q2)
3670 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(1-PART3667))
3680 GOSUB 1690
3690 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(PART5667)
3700 NEXT Q2
3710 NEXT Q1
3720 '
3730 '          PSEUDOAIRWAYS 3-6-8 AND 5-6-8
3740 '
3750 ALF = 60*PI    : BET = 120*PI      : GAM = 0*PI
3760 FOR Q1 = 1 TO ASN(3,6)
3770 FOR Q2 = 1 TO ASN(5,6)
3780 V1 = VEL(3,6,Q1) : V2 = VEL(5,6,Q2)
3790 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(1-PART5667))
3800 GOSUB 1500
3810 RC6 = RC6 + PCON*LAM(3,6)*PMF(3,6,Q1)*(1-PART3667)
3820 NEXT Q2
3830 NEXT Q1
3840 '
3850 BET =  60*PI
3860 FOR Q1 = 1 TO ASN(5,6)
3870 FOR Q2 = 1 TO ASN(3,6)
3880 V1 = VEL(5,6,Q1) : V2 = VEL(3,6,Q2)
3890 SEXP = (V2/LAM(3,6))/(PMF(3,6,Q2)*(1-PART3667))
3900 GOSUB 1500
```

```
3910 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(1-PART5667)
3920 NEXT Q2
3930 NEXT Q1
3940 '
3950 '          PSEUDOAIRWAYS 4-6-7 AND 4-6-8
3960 '
3970 ALF = 0*PI    : BET =  80*PI      : GAM = 105*PI
3980 FOR Q1 = 1 TO ASN(4,6)
3990 FOR Q2 = 1 TO ASN(4,6)
4000 V1 = VEL(4,6,Q1) : V2 = VEL(4,6,Q2)
4010 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(1-PART4667))
4020 GOSUB 1270
4030 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(PART4667)
4040 NEXT Q2
4050 NEXT Q1
4060 '
4070 BET =  25*PI
4080 FOR Q1 = 1 TO ASN(4,6)
4090 FOR Q2 = 1 TO ASN(4,6)
4100 V1 = VEL(4,6,Q1) : V2 = VEL(4,6,Q2)
4110 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(PART4667))
4120 GOSUB 1270
4130 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(1-PART4667)
4140 NEXT Q2
4150 NEXT Q1
4160 '
4170 '          PSEUDOAIRWAYS 4-6-7 AND 5-6-7
4180 '
4190 ALF = PI*95   : BET =  PI*15      : GAM = PI*0
4200 FOR Q1 = 1 TO ASN(4,6)
4210 FOR Q2 = 1 TO ASN(5,6)
4220 V1 = VEL(4,6,Q1) : V2 = VEL(5,6,Q2)
4230 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(PART5667))
4240 GOSUB 1500
4250 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(PART4667)
4260 NEXT Q2
4270 NEXT Q1
4280 '
4290 BET =  80*PI
4300 FOR Q1 = 1 TO ASN(5,6)
```

```
310 FOR Q2 = 1 TO ASN(4,6)
320 V1 = VEL(5,6,Q1) : V2 = VEL(4,6,Q2)
330 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(PART4667))
340 GOSUB 1500
350 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(PART5667)
360 NEXT Q2
370 NEXT Q1
380 '
390 '        PSEUDOAIRWAYS 4-6-7 AND 5-6-8
400 '
4410 ALF = PI*95    : BET =  PI*15     : GAM = PI*105
4420 FOR Q1 = 1 TO ASN(4,6)
4430 FOR Q2 = 1 TO ASN(5,6)
4440 V1 = VEL(4,6,Q1) : V2 = VEL(5,6,Q2)
4450 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(1-PART5667))
4460 GOSUB 1690
4470 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(PART4667)
4480 NEXT Q2
4490 NEXT Q1
4500 '
4510 BET =  25*PI
4520 FOR Q1 = 1 TO ASN(5,6)
4530 FOR Q2 = 1 TO ASN(4,6)
4540 V1 = VEL(5,6,Q1) : V2 = VEL(4,6,Q2)
4550 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(PART4667))
4560 GOSUB 1690
4570 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(1-PART5667)
4580 NEXT Q2
4590 NEXT Q1
4600 '
4610 '        PSEUDOAIRWAYS 4-6-8 AND 5-6-7
4620 '
4630 ALF = PI*95    : BET =  PI*120    : GAM = PI*105
4640 FOR Q1 = 1 TO ASN(4,6)
4650 FOR Q2 = 1 TO ASN(5,6)
4660 V1 = VEL(4,6,Q1) : V2 = VEL(5,6,Q2)
4670 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(PART5667))
4680 GOSUB 1690
4690 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(1-PART4667)
4700 NEXT Q2
4710 NEXT Q1
4720 '
4730 BET =  80*PI
4740 FOR Q1 = 1 TO ASN(5,6)
4750 FOR Q2 = 1 TO ASN(4,6)
4760 V1 = VEL(5,6,Q1) : V2 = VEL(4,6,Q2)
4770 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(1-PART4667))
4780 GOSUB 1690
4790 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(PART5667)
4800 NEXT Q2
```

```
4810 NEXT Q1
4820 '
4830 '          PSEUDOAIRWAYS 4-6-8 AND 5-6-8
4840 '
4850 ALF = PI*95    : BET =  PI*120     : GAM = PI*0
4860 FOR Q1 = 1 TO ASN(4,6)
4870 FOR Q2 = 1 TO ASN(5,6)
4880 V1 = VEL(4,6,Q1) : V2 = VEL(5,6,Q2)
4890 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(1-PART5667))
4900 GOSUB 1500
4910 RC6 = RC6 + PCON*LAM(4,6)*PMF(4,6,Q1)*(1-PART4667)
4920 NEXT Q2
4930 NEXT Q1
4940 '
4950 BET =  25*PI
4960 FOR Q1 = 1 TO ASN(5,6)
4970 FOR Q2 = 1 TO ASN(4,6)
4980 V1 = VEL(5,6,Q1) : V2 = VEL(4,6,Q2)
4990 SEXP = (V2/LAM(4,6))/(PMF(4,6,Q2)*(1-PART4667))
5000 GOSUB 1500
5010 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(1-PART5667)
5020 NEXT Q2
5030 NEXT Q1
5040 '
5050 '          PSEUDOAIRWAYS 5-6-7 AND 5-6-8
5060 '
5070 ALF = PI*0    : BET =  PI*15      : GAM = PI*105
5080 FOR Q1 = 1 TO ASN(5,6)
5090 FOR Q2 = 1 TO ASN(5,6)
5100 V1 = VEL(5,6,Q1) : V2 = VEL(5,6,Q2)
5110 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(1-PART5667))
5120 GOSUB 1270
5130 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(PART5667)
5140 NEXT Q2
5150 NEXT Q1
5160 '
5170 BET = 120*PI
5180 FOR Q1 = 1 TO ASN(5,6)
5190 FOR Q2 = 1 TO ASN(5,6)
5200 V1 = VEL(5,6,Q1) : V2 = VEL(5,6,Q2)
5210 SEXP = (V2/LAM(5,6))/(PMF(5,6,Q2)*(PART5667))
5220 GOSUB 1270
5230 RC6 = RC6 + PCON*LAM(5,6)*PMF(5,6,Q1)*(1-PART5667)
5240 NEXT Q2
5250 NEXT Q1
5260 '
5270 PRINT "RC6 IS: "; RC6
5280 '
5290 PRINT "RC IS: "; RC3 + RC6
5300 ROV = 0 : GOTO 5700
```

```
5310 '
5320 '
5330 ' SUBROUTINE TO COMPUTE OVERTAKE RATE WHEN TWO AIRCRAFT
5340 ' ENTER AIRWAY SEGMENT ON DIFFERENT TRACKS, OR ON SAME TRACK
5350 ' ALIGNED WITH THE SEGMENT (ie. BET = 0).
5360 '
5370 ' INPUTS: V1 = VELOCITY OF OVERTAKEN AIRCRAFT
5380 '         V2 = VELOCITY OF OVERTAKING AIRCRAFT
5390 '          L = LENGTH OF AIRWAY SEGMENT
5400 '          M = MINIMUM SEPARATION
5410 '       LAM2= TRAFFIC DENSITY OF OVERTAKING AIRCRAFT ON THIS AIRWAY
5420 '               (AIRCRAFT/HOUR).
5430 '
5440 IF V2 >= V1 THEN PNO = EXP((V2-V1)*L/(V1*(M-V2/LAM2)))

               ELSE PNO = 1
5450 RETURN
5460 '
5470 ' SUBROUTINE TO COMPUTE OVERTAKE RATE WHEN TWO AIRCRAFT
5480 ' ENTER AIRWAY SEGMENT ON THE SAME TRACK
5490 '
5500 ' INPUTS: V1 = VELOCITY OF OVERTAKEN AIRCRAFT
5510 '         V2 = VELOCITY OF OVERTAKING AIRCRAFT
5520 '          L = LENGTH OF AIRWAY SEGMENT
5530 '          M = MINIMUM SEPARATION
5540 '       LAM2 = TRAFFIC DENSITY OF OVERTAKING AIRCRAFT ON THIS AIRWAY
5550 '                (AIRCRAFT/HOUR).
5560 '        BET = ANGLE BETWEEN ARRIVAL TRACK
5570 '                  AND BACKWARD EXTENTION OF AIRWAY TRACK.
5580 '
5590 K = V2/V1
5600 IF K = 1 AND BET = 0 THEN PNO = 1 : RETURN
5610 IF BET = 0 THEN 5440
5620 ABET = K - COS(BET)
5630 KBET = (K^2 + 1 - 2*K*COS(BET))^(-1)
5640 SBET = ((-ABET*KBET)^2 + (1 - K*ABET*KBET)^2 - 2*(-ABET*KBET)*(1 -

         K*ABET*KBET)*COS(BET))^(.5)
5650 IF V1 >= V2  THEN IF  ABET*KBET < 0  THEN  PNO = 1

             ELSE  PNO = EXP(M*(1-1/SBET)/(V2/LAM2 - M)) ELSE 5670
5660 IF M + (V2-V1)*L/V1  >=  M/SBET     THEN

            D = M + (V2-V1)*L/V1     ELSE

               D = M/SBET
5670 IF V2 > V1  THEN  PNO = EXP((M-D)/(V2/LAM2 - M))
5680 RETURN
5690 '
5700 '
5710 ' OVERTAKE RATE ON AIRWAY SEGMENT 1-3
5720 '
5730 L = 50
5740 '
5750 FOR Q1 = 1 TO ASN(1,3)
5760       V1 = VEL(1,3,Q1) : LAM1 = LAM(1,3)*PMF(1,3,Q1) : PNOT = 1
5770       FOR Q2 = 1 TO ASN(1,3)
5780             V2 = VEL(1,3,Q2) : LAM2 = LAM(1,3)*PMF(1,3,Q2)
5790             GOSUB 5440
5800             PNOT = PNOT*PNO
```

```
5810        NEXT Q2
5820 '
5830        ROV = ROV + LAM1*(1 - PNOT)
5840 '
5850 NEXT Q1
5860 '
5870 '
5880 ' OVERTAKE RATE ON AIRWAY SEGMENT 2-3
5890 '
5900 L = 60
5910 '
5920 FOR Q1 = 1 TO ASN(2,3)
5930        V1 = VEL(2,3,Q1) : LAM1 = LAM(2,3)*PMF(2,3,Q1) : PNOT = 1
5940        FOR Q2 = 1 TO ASN(2,3)
5950              V2 = VEL(2,3,Q2) : LAM2 = LAM(2,3)*PMF(2,3,Q2)
5960              GOSUB 5440
5970              PNOT = PNOT*PNO
5980        NEXT Q2
5990 '
6000        ROV = ROV + LAM1*(1 - PNOT)
6010 '
6020 NEXT Q1
6030 '
6040 '
6050 ' OVERTAKE RATE ON AIRWAY 3-6
6060 '
6070 '        OVERTAKE RATE ON TRAFFIC ENTERING FROM 1-3
6080 '
6090 L = 50
6100 FOR Q1 = 1 TO ASN(1,3)
6110        PNOT = 1
6120        V1 = VEL(1,3,Q1) : LAM1 = LAM(1,3)*PMF(1,3,Q1)
6130        FOR Q2 = 1 TO ASN(2,3)
6140              V2 = VEL(2,3,Q2) : LAM2 = LAM(2,3)*PMF(2,3,Q2)
6150              GOSUB 5440
6160              PNOT = PNOT*PNO
6170        NEXT Q2
6180 '
6190        FOR Q3 = 1 TO ASN(1,3)
6200              V2 = VEL(1,3,Q3) : LAM2 = LAM(1,3)*PMF(1,3,Q3)
6210              BET = 45*PI
6220              GOSUB 5590
6230              PNOT = PNOT*PNO
6240        NEXT Q3
6250 '
6260        ROV = ROV + LAM1*(1 - PNOT)
6270 NEXT Q1
6280 '
6290 '
6300 '        OVERTAKE RATE ON TRAFFIC ENTERING FROM 2-3
```

```
6310 '
6320 FOR Q1 = 1 TO ASN(2,3)
6330       PNOT = 1
6340       V1 = VEL(2,3,Q1) : LAM1 = LAM(2,3)*PMF(2,3,Q1)
6350      FOR Q2 = 1 TO ASN(1,3)
6360            V2 = VEL(1,3,Q2) : LAM2 = LAM(1,3)*PMF(1,3,Q2)
6370            GOSUB 5440
6380            PNOT = PNOT*PNO
6390      NEXT Q2
6400 '
6410      FOR Q3 = 1 TO ASN(2,3)
6420            BET = 50*PI
6430            V2 = VEL(2,3,Q3) : LAM2 = LAM(2,3)*PMF(2,3,Q3)
6440            GOSUB 5590
6450            PNOT = PNOT*PNO
6460      NEXT Q3
6470 '
6480      ROV = ROV + LAM1*(1 - PNOT)
6490 NEXT Q1
6500 '
6510 '
6520 '
6530 '
6540 '
6550 ' OVERTAKE RATE ON AIRWAY 4-6
6560 '
6570 L = 65
6580 '
6590 FOR Q1 = 1 TO ASN(4,6)
6600      V1 = VEL(4,6,Q1) : LAM1 = LAM(4,6)*PMF(4,6,Q1) : PNOT = 1
6610      FOR Q2 = 1 TO ASN(4,6)
6620            V2 = VEL(4,6,Q2) : LAM2 = LAM(4,6)*PMF(4,6,Q2)
6630            GOSUB 5440
6640            PNOT = PNOT*PNO
6650      NEXT Q2
6660 '
6670      ROV = ROV + LAM1*(1 - PNOT)
6680 '
6690 NEXT Q1
6700 '
6710 '
6720 ' OVERTAKE RATE ON AIRWAY 5-6
6730 '
6740 L = 40
6750 '
6760 FOR Q1 = 1 TO ASN(5,6)
6770      V1 = VEL(5,6,Q1) : LAM1 = LAM(5,6)*PMF(5,6,Q1) : PNOT = 1
6780      FOR Q2 = 1 TO ASN(5,6)
6790            V2 = VEL(5,6,Q2) : LAM2 = LAM(5,6)*PMF(5,6,Q2)
6800            GOSUB 5440
```

```
6810             PNOT = PNOT*PNO
6820         NEXT Q2
6830 '
6840         ROV = ROV + LAM1*(1 - PNOT)
6850 '
6860 NEXT Q1
6870 '
6880 PRINT
6890 PRINT "THE OVERTAKE INTERVENTION RATE IS "; ROV
6900 PRINT
6910 PRINT "THE TOTAL INTERVENTION RATE IS " ; RC3 + RC6 + ROV
6920 END
6930 '
6940 '
6950 ' OVERTAKE RATE ON AIRWAY 6-7
6960 '
6970 L = 80
6980 '
6990 '       OVERTAKES ON AIRCRAFT FROM AIRWAY 3-6
7000 '
7010 FOR Q1 = 1 TO ASN(3,6)
7020      PNOT = 1 : BET = 45*PI
7030      V1 = VEL(3,6,Q1) : LAM1 = LAM(3,6)*PMF(3,6,Q1)*PART3667
7040      FOR Q2 = 1 TO ASN(3,6)
7050           V2 = VEL(3,6,Q2) : LAM2 = LAM(3,6)*PMF(3,6,Q2)*PART3667
7060           GOSUB 5590
7070           PNOT = PNOT*PNO
7080      NEXT Q2
7090 '
7100      FOR Q2 = 1 TO ASN(4,6)
7110           V2 = VEL(4,6,Q2)
7120           LAM2 = LAM(4,6)*PMF(4,6,Q2)*PART4667
7130           GOSUB 5440
7140           PNOT = PNOT*PNO
7150      NEXT Q2
7160 '
7170      FOR Q2 = 1 TO ASN(5,6)
7180           V2 = VEL(5,6,Q2)
7190           LAM2 = LAM(5,6)*PMF(5,6,Q2)*PART5667
7200           GOSUB 5440
7210           PNOT = PNOT*PNO
7220      NEXT Q2
7230 '
7240      ROV = ROV + LAM1*(1 - PNOT)
7250 NEXT Q1
7260 '
7270 '       OVERTAKES ON AIRCRAFT FROM AIRWAY 4-6
7280 '
7290 FOR Q1 = 1 TO ASN(4,6)
7300      PNOT = 1 : BET = 80*PI
```

```
7310       V1 = VEL(4,6,Q1) : LAM1 = LAM(4,6)*PMF(4,6,Q1)*PART4667
7320       FOR Q2 = 1 TO ASN(3,6)
7330             V2 = VEL(3,6,Q2) : LAM2 = LAM(3,6)*PMF(3,6,Q2)*PART3667
7340             GOSUB 5440
7350             PNOT = PNOT*PNO
7360       NEXT Q2
7370 '
7380       FOR Q2 = 1 TO ASN(4,6)
7390             V2 = VEL(4,6,Q2)
7400             LAM2 = LAM(4,6)*PMF(4,6,Q2)*PART4667
7410             GOSUB 5590
7420             PNOT = PNOT*PNO
7430       NEXT Q2
7440 '
7450       FOR Q2 = 1 TO ASN(5,6)
7460             V2 = VEL(5,6,Q2)
7470             LAM2 = LAM(5,6)*PMF(5,6,Q2)*PART5667
7480             GOSUB 5440
7490             PNOT = PNOT*PNO
7500       NEXT Q2
7510 '
7520       ROV = ROV + LAM1*(1 - PNOT)
7530 NEXT Q1
7540 '
7550 '        OVERTAKES ON AIRCRAFT FROM AIRWAY 5-6
7560 '
7570 FOR Q1 = 1 TO ASN(5,6)
7580       PNOT = 1 : BET = 15*PI
7590       V1 = VEL(5,6,Q1) : LAM1 = LAM(5,6)*PMF(5,6,Q1)*PART5667
7600       FOR Q2 = 1 TO ASN(3,6)
7610             V2 = VEL(3,6,Q2) : LAM2 = LAM(3,6)*PMF(3,6,Q2)*PART3667
7620             GOSUB 5440
7630             PNOT = PNOT*PNO
7640       NEXT Q2
7650 '
7660       FOR Q2 = 1 TO ASN(4,6)
7670             V2 = VEL(4,6,Q2)
7680             LAM2 = LAM(4,6)*PMF(4,6,Q2)*PART4667
7690             GOSUB 5440
7700             PNOT = PNOT*PNO
7710       NEXT Q2
7720 '
7730       FOR Q2 = 1 TO ASN(5,6)
7740             V2 = VEL(5,6,Q2)
7750             LAM2 = LAM(5,6)*PMF(5,6,Q2)*PART5667
7760             GOSUB 5590
7770             PNOT = PNOT*PNO
7780       NEXT Q2
7790 '
7800       ROV = ROV + LAM1*(1 - PNOT)
```

```
7810 NEXT Q1
7820 '
7830 '
7840 '
7850 ' OVERTAKE RATE ON AIRWAY 6-8
7860 '
7870 L = 50
7880 '
7890 '        OVERTAKES ON AIRCRAFT FROM AIRWAY 3-6
7900 '
7910 FOR Q1 = 1 TO ASN(3,6)
7920      PNOT = 1 : BET = 60*PI
7930      V1 = VEL(3,6,Q1) : LAM1 = LAM(3,6)*PMF(3,6,Q1)*(1-PART3667)
7940      FOR Q2 = 1 TO ASN(3,6)
7950           V2 = VEL(3,6,Q2) : LAM2 = LAM(3,6)*PMF(3,6,Q2)*(1-PART3667)
7960           GOSUB 5590
7970           PNOT = PNOT*PNO
7980      NEXT Q2
7990 '
8000      FOR Q2 = 1 TO ASN(4,6)
8010           V2 = VEL(4,6,Q2)
8020           LAM2 = LAM(4,6)*PMF(4,6,Q2)*(1-PART4667)
8030           GOSUB 5440
8040           PNOT = PNOT*PNO
8050      NEXT Q2
8060 '
8070      FOR Q2 = 1 TO ASN(5,6)
8080           V2 = VEL(5,6,Q2)
8090           LAM2 = LAM(5,6)*PMF(5,6,Q2)*(1-PART5667)
8100           GOSUB 5440
8110           PNOT = PNOT*PNO
8120      NEXT Q2
8130 '
8140      ROV = ROV + LAM1*(1 - PNOT)
8150 NEXT Q1
8160 '
8170 '        OVERTAKES ON AIRCRAFT FROM AIRWAY 4-6
8180 '
8190 FOR Q1 = 1 TO ASN(4,6)
8200      PNOT = 1 : BET = 25*PI
8210      V1 = VEL(4,6,Q1) : LAM1 = LAM(4,6)*PMF(4,6,Q1)*(1-PART4667)
8220      FOR Q2 = 1 TO ASN(3,6)
8230           V2 = VEL(3,6,Q2) : LAM2 = LAM(3,6)*PMF(3,6,Q2)*(1-PART3667)
8240           GOSUB 5440
8250           PNOT = PNOT*PNO
8260      NEXT Q2
8270 '
8280      FOR Q2 = 1 TO ASN(4,6)
8290           V2 = VEL(4,6,Q2)
8300           LAM2 = LAM(4,6)*PMF(4,6,Q2)*(1-PART4667)
```

```
8310            GOSUB 5590
8320            PNOT = PNOT*PNO
8330       NEXT Q2
8340 '
8350       FOR Q2 = 1 TO ASN(5,6)
8360            V2 = VEL(5,6,Q2)
8370            LAM2 = LAM(5,6)*PMF(5,6,Q2)*(1-PART5667)
8380            GOSUB 5440
8390            PNOT = PNOT*PNO
8400       NEXT Q2
8410 '
8420       ROV = ROV + LAM1*(1 - PNOT)
8430 NEXT Q1
8440 '
8450 '        OVERTAKES ON AIRCRAFT FROM AIRWAY 5-6
8460 '
8470 FOR Q1 = 1 TO ASN(5,6)
8480       PNOT = 1 : BET = 120*PI
8490       V1 = VEL(5,6,Q1) : LAM1 = LAM(5,6)*PMF(5,6,Q1)*(1-PART5667)
8500       FOR Q2 = 1 TO ASN(3,6)
8510            V2 = VEL(3,6,Q2) : LAM2 = LAM(3,6)*PMF(3,6,Q2)*(1-PART3667)
8520            GOSUB 5440
8530            PNOT = PNOT*PNO
8540       NEXT Q2
8550 '
8560       FOR Q2 = 1 TO ASN(4,6)
8570            V2 = VEL(4,6,Q2)
8580            LAM2 = LAM(4,6)*PMF(4,6,Q2)*(1-PART4667)
8590            GOSUB 5440
8600            PNOT = PNOT*PNO
8610       NEXT Q2
8620 '
8630       FOR Q2 = 1 TO ASN(5,6)
8640            V2 = VEL(5,6,Q2)
8650            LAM2 = LAM(5,6)*PMF(5,6,Q2)*(1-PART5667)
8660            GOSUB 5590
8670            PNOT = PNOT*PNO
8680       NEXT Q2
8690 '
8700       ROV = ROV + LAM1*(1 - PNOT)
8710 NEXT Q1
8720 '
8730 PRINT
8740 PRINT "THE OVERTAKE INTERVENTION RATE IS "; ROV
8750 PRINT
8760 PRINT "THE TOTAL INTERVENTION RATE IS " ; RC3 + RC6 + ROV
8770 END
```

PROGRAM 3

```
10 '                           PROGRAM 3
20 '                  SIMPLE OVERTAKE SIMULATION
30 '
40 '      SUBSTITUTE LAM FOR S INPUT
50 '
60 '
70 '       INITIALIZE
80 '
90 DIM V(20), DAT(4,30), PMF(20)
100 INPUT "LENGTH OF AIRWAY (NM) "; L
110 INPUT "NUMBER OF AIRSPEEDS "; N
120 INPUT "VALUE OF FIRST AIRSPEED (KTS) "; V(1)
130 FOR Q1 = 1 TO N-1
140      INPUT "NEXT " ; V(Q1+1)
150 NEXT Q1
160 '
170 INPUT "PROBABILITY OF FIRST AIRSPEED "; PMF(1)
180 FOR Q1 = 1 TO N-1
190      INPUT "NEXT PMF "; PMF(Q1+1)
200 NEXT Q1
210 '
220 INPUT "TRAFFIC DENSITY (AIRCRAFT/HR) "; LAM
230 INPUT "LENGTH OF RUN (HOURS)"; FINTIME
240 RANDOMIZE TIMER
250 '
260 INPUT "MINIMUM SEPARATION DESIRED"; M
270 FOR Q = 1 TO 30
280      DAT(1,Q)=L*10^10:DAT(2,Q)=0:DAT(3,Q)=10*FINTIME:DAT(4,Q)=11*FINTIME
290 NEXT Q
300 '
310 CLOCK=0 : NUM=1 : OVER=0
320 GOSUB 800
330 DAT(1,1)=L : DAT(2,1)=AS
340 NXCRITIM=DIST/AS : DAT(3,1)=DIST/AS : DAT(4,1)=(DIST+L)/AS
350 '
360 '       AIRCRAFT ARRIVES AT ENTRY POINT
370 '
380 IF NUM=1 THEN 520
390 IF (DAT(2,NUM)-DAT(2,NUM-1))*(DAT(4,NUM-1)-CLOCK)<DAT(1,NUM)-DAT(1,NUM-1)-M
       THEN 520
400 OVER = OVER + 1 : QR = NUM
410 GOSUB 950
420 GOSUB 800
430 NUM = NUM+1 : DAT(1,NUM)=L+DIST : DAT(2,NUM)=AS
440 DAT(3,NUM)=NXCRITIM + DIST/AS : DAT(4,NUM)=DAT(3,NUM)+L/AS
450 FOR Q = 1 TO NUM - 1
460      DAT(1,Q)=DAT(1,Q)-DAT(2,Q)*(NXCRITIM - CLOCK)
470 NEXT Q
480 CLOCK = NXCRITIM : IF CLOCK >= FINTIME THEN 1110
490 GOSUB 1040
500 IF NXCRITIM=DAT(3,NUM) THEN 360 ELSE 630
```

```
510 '
520 GOSUB 800
530 NUM=NUM+1
540 DAT(1,NUM) = L+DIST : DAT(2,NUM)=AS
550 DAT(3,NUM)=NXCRITIM+DIST/AS : DAT(4,NUM)=DAT(3,NUM)+L/AS
560 FOR Q = 1 TO NUM - 1
570      DAT(1,Q)=DAT(1,Q)-DAT(2,Q)*(NXCRITIM - CLOCK)
580 NEXT Q
590 CLOCK = NXCRITIM : IF CLOCK)=FINTIME THEN 1110
600 GOSUB 1040
610 IF NXCRITIM=DAT(3,NUM) THEN 360 ELSE 630
620 '
630 '      AIRCRAFT EXITS AIRWAY
640 '
650 QR=1
660 GOSUB 950
670 '
680 FOR Q=1 TO NUM
690      DAT(1,Q)=DAT(1,Q)-(NXCRITIM-CLOCK)*DAT(2,Q)
700 NEXT Q
710 CLOCK=NXCRITIM : IF CLOCK)=FINTIME THEN 1110
720 GOSUB 1040
730 IF NXCRITIM=DAT(4,1) THEN 630 ELSE 360
740 '
750 '
760 '
770 '       SUBROUTINES
780 '
790 '
800 '    PICK NEXT AIRSPEED
810 '
820 X=RND : REF=PMF(1) : IF CLOCK)=FINTIME THEN 1110
830 FOR Q=1 TO N
840      IF X(=REF THEN AS=V(Q) : PM =PMF(Q) : GOTO 870
850      REF=REF+PMF(Q+1)
860 NEXT Q
870 '
880 '
890 '     PICK DISTANCE TO NEXT ENTERING AIRCRAFT
900 '
910 X=RND
920 DIST = (M - AS/(LAM))*LOG(1-X) + M
930 RETURN
940 '
950 '      REMOVE COLUMN
960 '
970 FOR Q=QR TO NUM
980      DAT(1,Q)=DAT(1,Q+1) :  DAT(2,Q)=DAT(2,Q+1)
990      DAT(3,Q)=DAT(3,Q+1) :  DAT(4,Q)=DAT(4,Q+1)
1000 NEXT Q
```

```
1010 NUM=NUM-1
1020 RETURN
1030 '
1040 '    PICK NEXT CRITICAL TIME
1050 '
1060 IF DAT(4,1)(=DAT(3,NUM) THEN NXCRITIM=DAT(4,1) ELSE NXCRITIM=DAT(3,NUM)
1070 '
1080 RETURN
1090 '
1100 '
1110 PRINT "OVERTAKE RATE = "; OVER/CLOCK
1120 '
1130 PRINT
1140 INPUT "ANOTHER RUN (YES = 1, NO = 0) "; X
1150 IF X = 1 THEN 270 ELSE END
```

PROGRAM 4

```
10 '          PROGRAM 4  (VERSION 3)
20 '
30 '     SIMULATION OF SIMPLE INTERSECTION - FIGURES SAMPLE VARIATION
40 '
50 '
60 '   INITIALIZE VARIABLES AND INPUT DATA
70 '
80 DIM DAT(2,40), NXCRITAC(2),INTV(200),VAR(200)
100 '
110 INPUT "ANGLE BETWEEN AIRWAYS "; ALP : ALPHA=ALP*3.14159/180
120 INPUT "AIRSPEED ON AIRWAY #1"; V1
130 INPUT "AIRSPEED ON AIRWAY #2"; V2
140 INPUT "MINIMUM SEPARATION DISTANCE "; M
150 INPUT "EXPECTED SEPARATION ON AIRWAY #1 "; S1
160 INPUT "EXPECTED SEPARATION ON AIRWAY #2 "; S2
170 INPUT "DISTANCE FROM ENTRY TO INTXN ON AIRWAY #1"; L1
180 INPUT "DISTANCE FROM ENTRY TO INTXN ON AIRWAY #2"; L2
190  FINTIME = 9
200 LAR = 10*(L1+L2)*FINTIME
210 RANDOMIZE TIMER
215 FOR T=1 TO 100
218 CLOCK=0 : NUM1=1 : NUM2=1 : INTER=0
220 '
230 FOR Q1 = 1 TO 2
240     FOR Q2 = 1 TO 40
250          DAT(Q1,Q2)=LAR
270     NEXT Q2
280 NEXT Q1
290 '
300 CDF1=RND : CDF2=RND
310 X1=(M-S1)*LOG(1-CDF1)+M
320 X2=(M-S2)*LOG(1-CDF2)+M
330 DAT(1,1)=X1+L1 : DAT(2,1)=X1/V1
340 DAT(1,21)=X2+L2 : DAT(2,21)=X2/V2
350 IF DAT(2,1)<=DAT(2,21) THEN NXCRITAC(1)=1 : NXCRITAC(2)=1 ELSE NXCRITAC(1)=2
        : NXCRITAC(2)=21
370 IF NXCRITAC(1)=1 THEN NXCRITIM = DAT(2,1) ELSE NXCRITIM=DAT(2,21)
380 '
390 '
400 K=V2/V1 : IF K = 1 AND ALPHA = 0 THEN C=1 : GOTO 430
410 A=K-COS(ALPHA) : CAPK=1/(K^2+1-2*K*COS(ALPHA))
420 C=((CAPK*A)^2*(1+K^2)+1+2*CAPK*A*(COS(ALPHA)-K-CAPK*K*A*COS(ALPHA)))^(-.5)
430 CRITDIST1= M*C
440 K=V1/V2 : IF K = 1 AND ALPHA = 0 THEN C=1 : GOTO 470
450 A=K-COS(ALPHA) : CAPK=1/(K^2+1-2*K*COS(ALPHA))
460 C=((CAPK*A)^2*(1+K^2)+1+2*CAPK*A*(COS(ALPHA)-K-CAPK*K*A*COS(ALPHA)))^(-.5)
470 CRITDIST2= M*C
480 '
500 '      AIRCRAFT ARRIVING AT ENTRY POINT
510 '
520 FOR Q1=1 TO NUM1
530      DAT(1,Q1)=DAT(1,Q1)-V1*(NXCRITIM-CLOCK)
540 NEXT Q1
550 FOR Q1 = 1 TO NUM2
560      DAT(1,Q1+20)=DAT(1,Q1+20)-V2*(NXCRITIM-CLOCK)
570 NEXT Q1
580 '
590 IF NXCRITAC(1)=2 THEN 700
595 DAT(2,NUM1)=NXCRITIM+L1/V1
600 CDF1=RND : X1=(M-S1)*LOG(1-CDF1)+M
610 DAT(1,NUM1+1)=X1+L1: DAT(2,NUM1+1)=X1/V1 + NXCRITIM
620 NUM1=NUM1+1 : IF NUM1>20 THEN PRINT "TOO MANY A/C ON A/W 1" : GOTO 2100
625 GOTO 800
630 '
700 CDF2=RND : X2=(M-S2)*LOG(1-CDF2)+M : DAT(2,NUM2+20)=NXCRITIM+L2/V2
```

```
710 DAT(1,20+NUM2+1)=X2+L2: DAT(2,20+NUM2+1)=X2/V2 + NXCRITIM
720 NUM2=NUM2+1 : IF NUM2>20 THEN PRINT "TOO MANY A/C ON A/W 2" : GOTO 2100
730 '
800 IF NUM1=1 THEN 840
810 FOR Q1 =1 TO NUM1-1
820      DAT(2,Q1)=DAT(1,Q1)/V1 + NXCRITIM
830 NEXT Q1
840 IF NUM2=1 THEN 880
850 FOR Q1 =1 TO NUM2-1
860      DAT(2,Q1+20)=DAT(1,Q1+20)/V2 + NXCRITIM
870 NEXT Q1
880 '
890 CLOCK=NXCRITIM
900 IF CLOCK>=FINTIME THEN 2000
910 '
1000 '      PICKING NEXT CRITICAL TIME AND CRITICAL AIRCRAFT
1010 '
1020 NXCRITIM=DAT(2,1)
1030 FOR Q1=1 TO NUM1
1040      IF DAT(2,Q1)<=NXCRITIM THEN NXCRITIM=DAT(2,Q1):NXCRITAC(1)=1:
                NXCRITAC(2)=Q1
1060 NEXT Q1
1070 FOR Q2=1 TO NUM2
1080      IF DAT(2,Q2+20)<=NXCRITIM THEN NXCRITIM=DAT(2,Q2+20):NXCRITAC(1)=2:
                       NXCRITAC(2)=Q2+20
1100 NEXT Q2
1110 '
1120 IF NXCRITAC(2)=NUM1 OR NXCRITAC(2)=NUM2+20 THEN 500
1125 IF NXCRITAC(2)<>1 AND NXCRITAC(2)<>21 THEN PRINT "ERROR AT 1000":GOTO 2100
1130 '
1140 '
1500 '      AIRCRAFT ARRIVING AT INTERSECTION
1510 '
1530 '
1540 IF NXCRITAC(1)=2 THEN 1600
1550 '
1560 FOR Q1=1 TO NUM1-1
1570      DAT(1,Q1)=DAT(1,Q1+1) : DAT(2,Q1)=DAT(2,Q1+1)
1580 NEXT Q1
1590 DAT(1,NUM1)=LAR : DAT(2,NUM1)=LAR :NUM1=NUM1-1 : GOTO 1700
1595 '
1600 FOR Q2=1 TO NUM2-1
1610      DAT(1,Q2+20)=DAT(1,Q2+21) : DAT(2,Q2+20)=DAT(2,Q2+21)
1630 NEXT Q2
1640 DAT(1,NUM2+20)=LAR : DAT(2,NUM2+20)=LAR :NUM2=NUM2-1
1650 '
1660 '
1700 FOR Q1=1 TO NUM1-1
1710      DAT(1,Q1)=DAT(1,Q1)-V1*(NXCRITIM-CLOCK)
1720      DAT(2,Q1)=DAT(1,Q1)/V1 + NXCRITIM
1730 NEXT Q1
1740 DAT(1,NUM1)=DAT(1,NUM1)-V1*(NXCRITIM-CLOCK)
1750 '
1760 FOR Q2=1 TO NUM2-1
1770      DAT(1,Q2+20)=DAT(1,Q2+20)-V2*(NXCRITIM-CLOCK)
1780      DAT(2,Q2+20)=DAT(1,Q2+20)/V2 + NXCRITIM
1790 NEXT Q2
1800 DAT(1,NUM2+20)=DAT(1,NUM2+20)-V2*(NXCRITIM-CLOCK)
1810 '
1820 IF NXCRITAC(1)=1 THEN IF NUM2>0 AN
1820 IF NXCRITAC(1)=1 THEN IF NUM2>0 AND DAT(1,21)<= CRITDIST1 AND NXCRITIM>1
               THEN INTER=INTER+1
1825 IF NXCRITAC(1)=2 THEN IF NUM1>0 AND DAT(1,1)<=CRITDIST2  AND NXCRITIM>1
               THEN INTER=INTER+1
1830 '
1840 CLOCK=NXCRITIM
```

```
[illegible] IF CLOCK>FINTIME THEN 2000
1860 '
1870 GOTO 1000
1880 '
2000 PRINT : PRINT : PRINT "INTERVENTION RATE = "; INTER/(CLOCK-1)
2001 PRINT:PRINT "T = ";T
2002 INTV(T) = INTER
2010 NEXT T
2020 AVG = 0
2021 FOR Q=1 TO 100
2022      AVG=AVG+INTV(Q)
2023 NEXT Q
2024 AVG=AVG/100
2030 FOR Q=1 TO 100
2031      VAR(Q) = (INTV(Q)-AVG)^2
2032 NEXT Q
2039 SAMVAR=0
2040 FOR Q=1 TO 100
2041 SAMVAR = SAMVAR + VAR(Q)
2042 NEXT Q
2050 SAMVAR = SAMVAR/99
2060 PRINT:PRINT:PRINT "SAMPLE VARIANCE = "; SAMVAR
2100 END
```

PROGRAM 5

```
10 m=5 \ r=1.0352 \ l=2880 \ ai=30
20 dim b(577),  f(577), p(577), q(577)
30 !
40 !     compute d
50 !
60 a=1-cos(ai*3.14159/180)
70 k=1/(2*a)
80 c=(2*(k*a)^2+1+2*k*a*(cos(ai*3.14159/180)-1-k*a*cos(ai*3.14159/180)))^(-0.5)
90 d=c*m
100 print "d = ";d
110 !
180 !
190 !      compute p
200 !
210 p(0)=0 \ p(576)=1
220 for x=1 to 575
230 p(x)=1-((l-m*x)/l)*exp((m-d)*x/(l-m*x))
238      ! print "x= ";x ;"p(x) = ";p(x)
240 next x
245 print "PCON computed"
250 !
260 !      compute q    ( = P(N)    )
270 !
280 q(0)=exp(-2880/55)
290 for i=1 to 146
300 q(i) = exp(-(2880-5*i)/55)
310      for j=1 to i
320      q(i) = q(i)*((2880-5*i)/55)/j
330      next j
333 if q(i)< 10e-15 then 336
334 goto 340
336 q(i)=0
340 next i
350 print "P(N) computed"
410 !
420 !      compute sigma
430 !
440 for i=0 to 146
450 for j=0 to 146
460 !
462 if i=0 then 490
464 if j=0 then 490
470 s=s+q(i)*q(j)*((2*i*p(j)-8*r)^2 + 2*i*p(j)*(1-p(j)))
475 ! print i,j,s,q(i),q(j)
480 !
490 next j
500 next i
510 !
520 print "sigma = ";s
530 end
```

PROGRAM 6

```
10  '
20  '
30  '
40  '     THIS PROGRAM COMPARES THE DIFFERENCE BETWEEN THE PDFs FOR
50  '         INTERARRIVAL DISTANCES (DELAYED VS SIMPLE NEG EXP DIST) WHEN
60  '         THE ARRIVAL OF INTEREST IS OF PROBABILITY P
70  '
80  '
90  '
100  INPUT "EXPECTED DISTANCE BETWEEN ARRIVALS ";S
110  INPUT "P = ";P
120  INPUT "RUN LIMIT IN 5 MILE INCREMENTS ";LIMIT
130 PRINT:PRINT:PRINT:PRINT
140  PRINT "DISTANCE (NM)", "ACTUAL PDF", "APPROX PDF", "   % ERR", "% TOT ERR"
150  PRINT "-------------", "----------", "----------", "   -----", "---------"
160  PRINT
170 LAM=(S-M)^(-1): ACTSUM=0:  ERRSUM=0:  M=5:  DELTA=.1
180 FOR INTERVAL=0 TO LIMIT
190     FOR Q=0 TO M/DELTA - 1
200     X=M*INTERVAL + Q*DELTA
210     IF INTERVAL=0 THEN APPROX=0:ACTUAL=0: GOTO 310
220     APPROX = (LAM*P)*(EXP((M-X)*LAM*P))
230     ACTUAL=0
240           FOR NUM=1 TO INTERVAL
250           Y = X - NUM*M
260           IF NUM<3 THEN FACT=1 ELSE FACT = FACT*(NUM-1)
270           ACTUAL=ACTUAL+P*(1-P)^(NUM-1)*((LAM)^(NUM))*(Y^(NUM-1))*
                          (EXP(-LAM*Y))/FACT
280          NEXT NUM
290 ACTSUM=ACTSUM+ACTUAL: ERRSUM=ERRSUM+ACTUAL-APPROX
300 IF ACTSUM=0 THEN DEVTOT=0 ELSE DEVTOT = ERRSUM/ACTSUM
310   IF ABS(Q/50 - INT(Q/50)) > .01 THEN 350
320  ' PRINT
330 IF ACTUAL=0 THEN PRINT X,ACTUAL,APPROX: GOTO 350
340 PRINT X, ACTUAL, APPROX, (ACTUAL-APPROX)/ACTUAL, DEVTOT
350    NEXT Q
360 NEXT INTERVAL
370 END
```

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